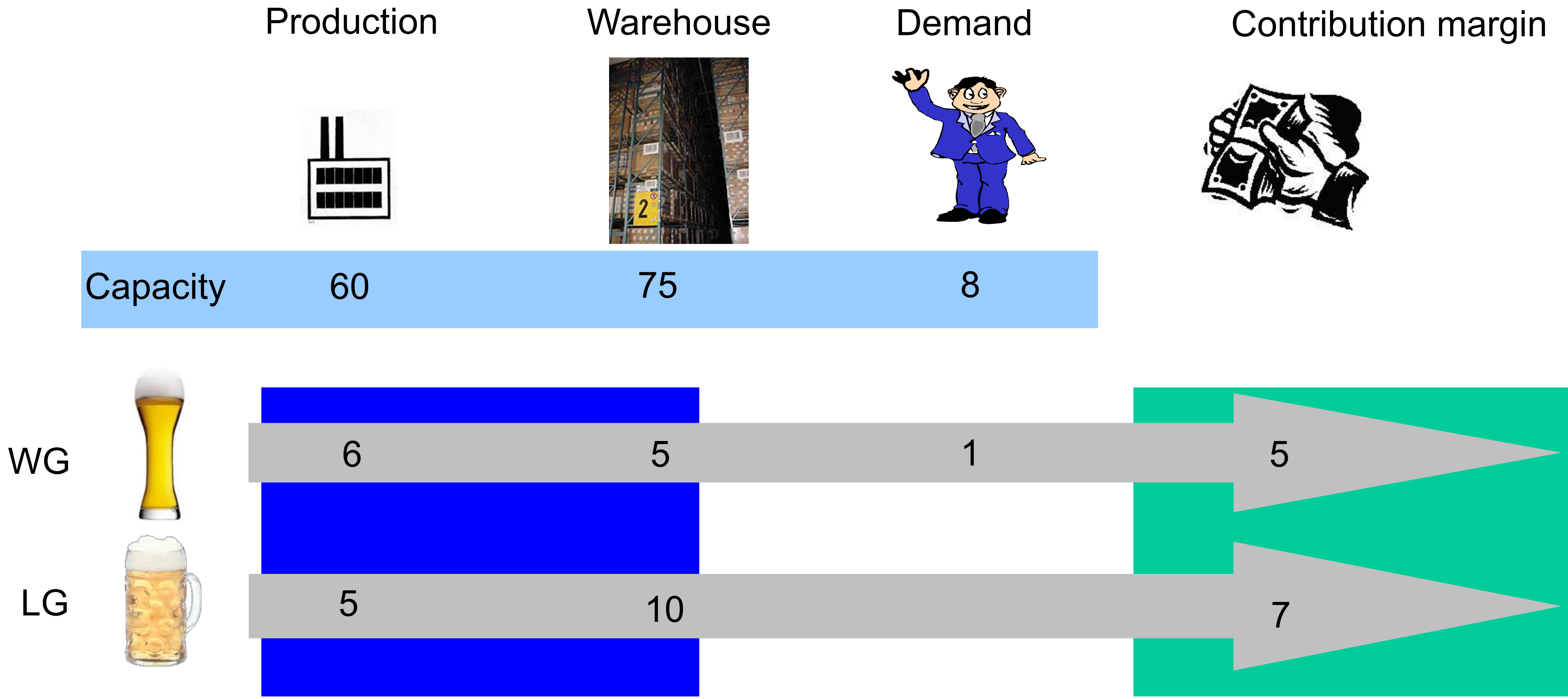
3 Linear Programming

3.1

Linear Programming and Graphical Solution

Production Planning Example



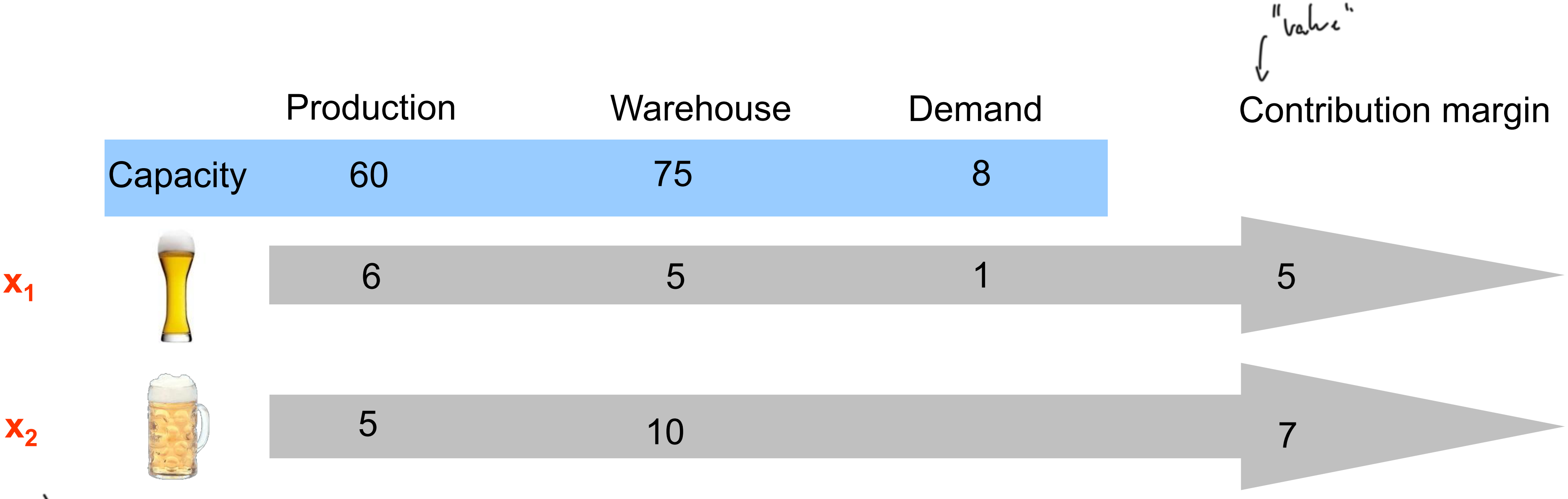
	Production	Warehouse	Demand	Contribution margin
Capacity	60	75	8	
WG	6	5	1	5
LG	5	10		7

Phases of an OR-Application

Problem → Modelling → Solving → Sensitivity Analysis → Decision Making

3.1.2 Modelling as Linear Program

Decision Variables



"value"

		Production	Warehouse	Demand	Contribution margin
	Capacity	60	75	8	
x_1		6	5	1	5
x_2		5	10		7

\# of ... beer glasses to be produced

Maximize total value z: $\max z = 5x_1 + 7x_2$

Production constraint: $6x_1 + 5x_2 \leq 60$

Warehouse constraint: $5x_1 + 10x_2 \leq 75$

Demand constraint: $x_1 \leq 8$

Objective Function

		Production	Warehouse	Demand	**Contribution margin**
Capacity		60	75	8	
x_1		6	5	1	**5**
x_2		5	10		**7**

Production Facility Constraint

		Production	Warehouse	Demand	Contribution margin
	Capacity	**60**	75	8	
x_1		**6**	5	1	5
x_2		5	10		7

Warehouse Constraint

		Production	**Warehouse**	Demand	Contribution margin
Capacity		60	**75**	8	
x_1		6	**5**	1	5
x_2		5	**10**		7

Demand Constraint

		Production	Warehouse	**Demand**	Contribution margin
	Capacity	60	75	**8**	
x_1		6	5	**1**	5
x_2		5	10		7

Linear Program

Maximize $z = \overbrace{5\,x_1 + 7\,x_2}^{c}$ (3.1) Objective function

subject to

$$\overbrace{\begin{matrix}6\,x_1 & + & 5\,x_2\\ 5\,x_1 & + & 10x_2\\ 1\,x_1 & & \end{matrix}}^{A}\begin{matrix}\leq\\ \leq\\ \leq\end{matrix}\overbrace{\begin{matrix}60\\ 75\\ 8\end{matrix}}^{b}\qquad\begin{matrix}\text{(3.2) Production constraint}\\ \text{(3.3) Warehouse constraint}\\ \text{(3.4) Market constraint}\end{matrix}$$

$1\,x_1 \geq 0$ (3.5 a) Non-negativity constraint

$1\,x_2 \geq 0$ (3.5 b) Non-negativity constraint

$$\max\ c^T x$$
$$\text{s.t.}\ A \cdot x \leq b$$

$$c = \begin{pmatrix}5\\ 7\end{pmatrix} \qquad b = \begin{pmatrix}60\\ 75\\ 8\end{pmatrix}$$

$$x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \qquad A = \begin{pmatrix}6 & 5\\ 5 & 10\\ 1 & 0\end{pmatrix}$$

3.1.3 Graphical Representation of the LP

Good for 2 vars, possible for 3, not useful for more

Linear Program

Maximize $z = 5x_1 + 7x_2$ (3.1) Objective function

subject to

$6x_1 + 5x_2 \leq 60$ (3.2) Production constraint

$5x_1 + 10x_2 \leq 75$ (3.3) Warehouse constraint

$1x_1 \leq 8$ (3.4) Market constraint

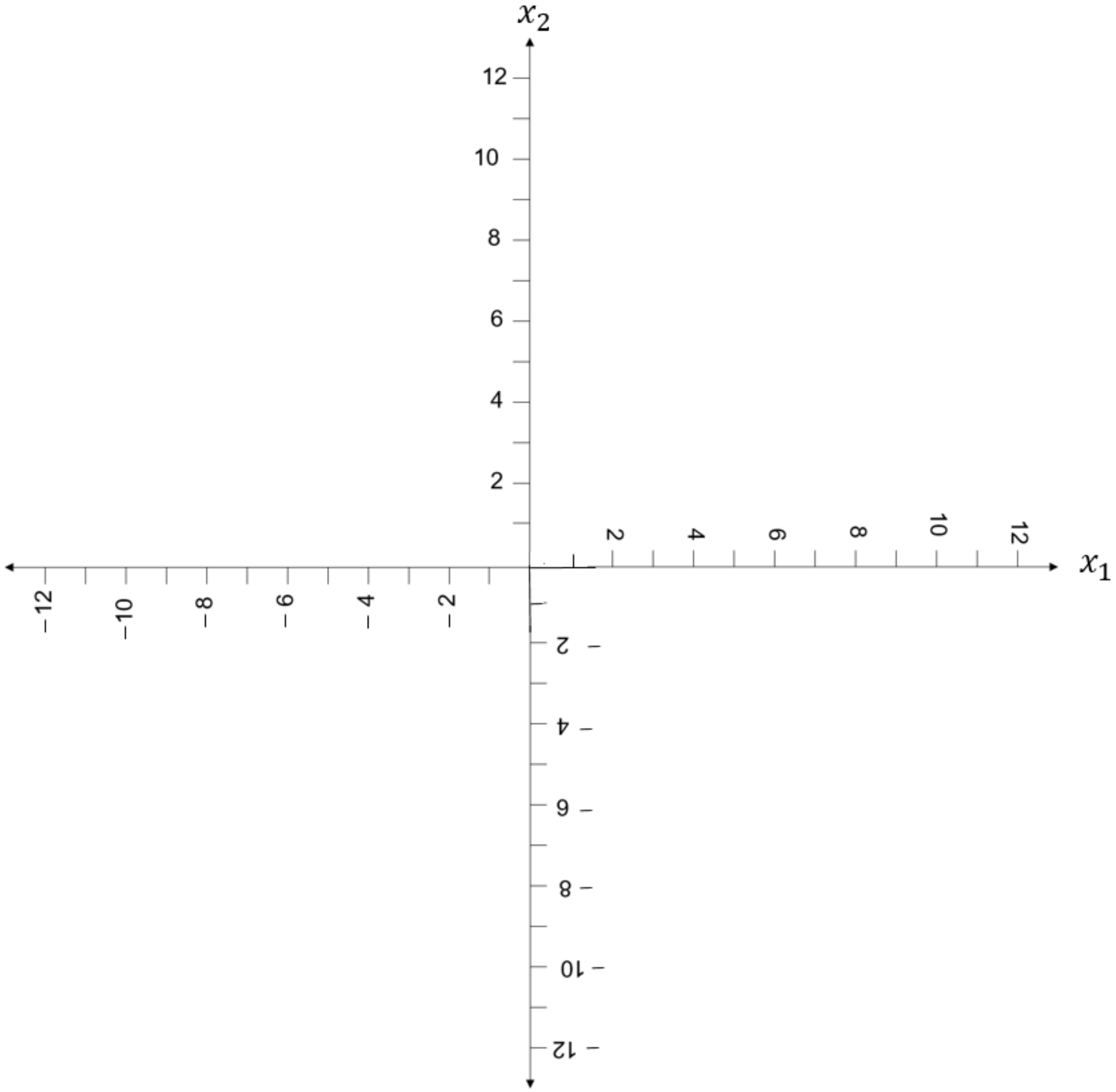
$1x_1 \geq 0$ (3.5 a) Non-negativity constraint

$1x_2 \geq 0$ (3.5 b) Non-negativity constraint

$x_1 \in \mathbb{R}$

$x_2 \in \mathbb{R}$

Solution Points $x \in \mathbb{R}^2$



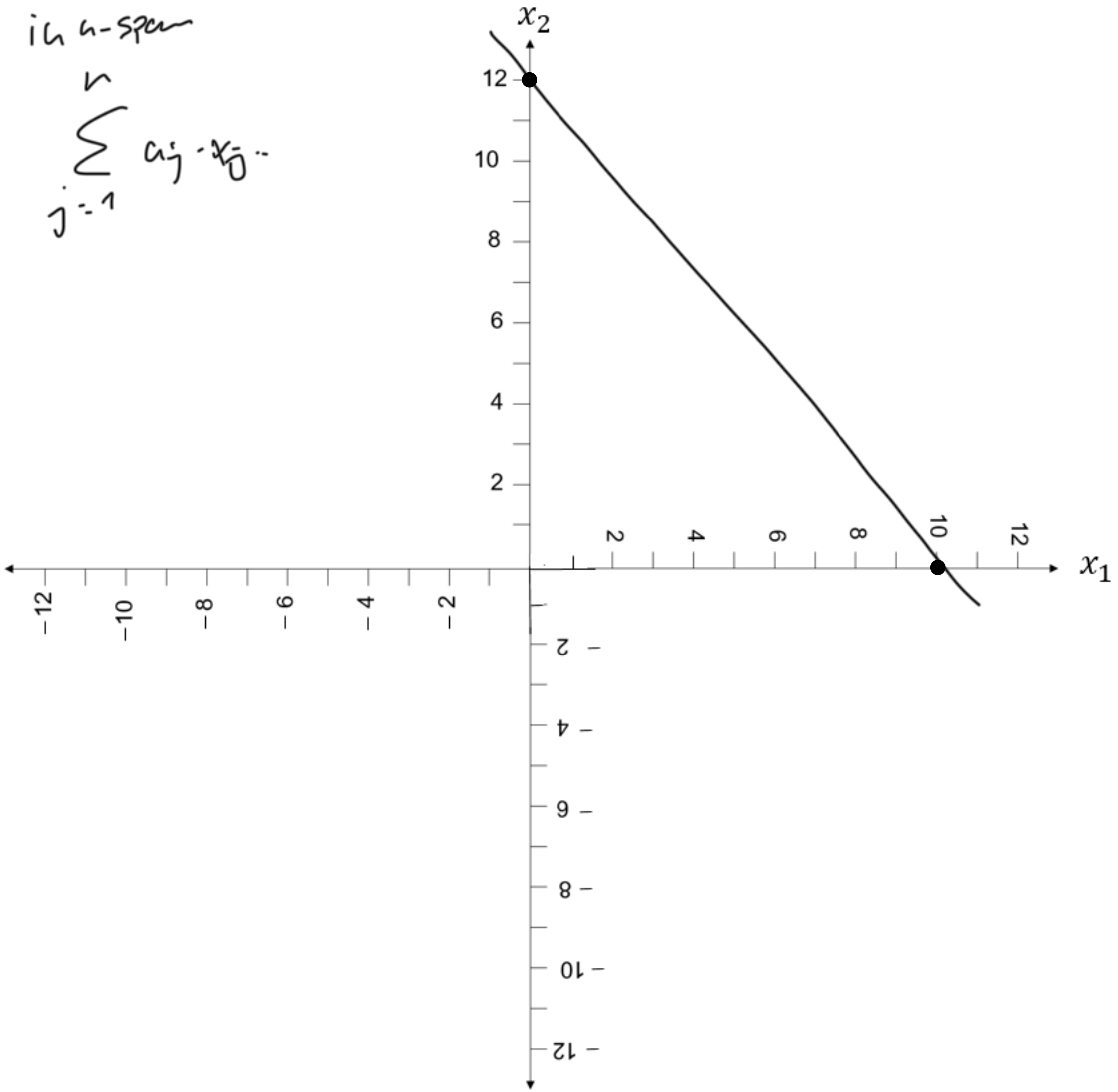
$x_1 \in \mathbb{R}$

$x_2 \in \mathbb{R}$

Hyperplane from the Production Constraint

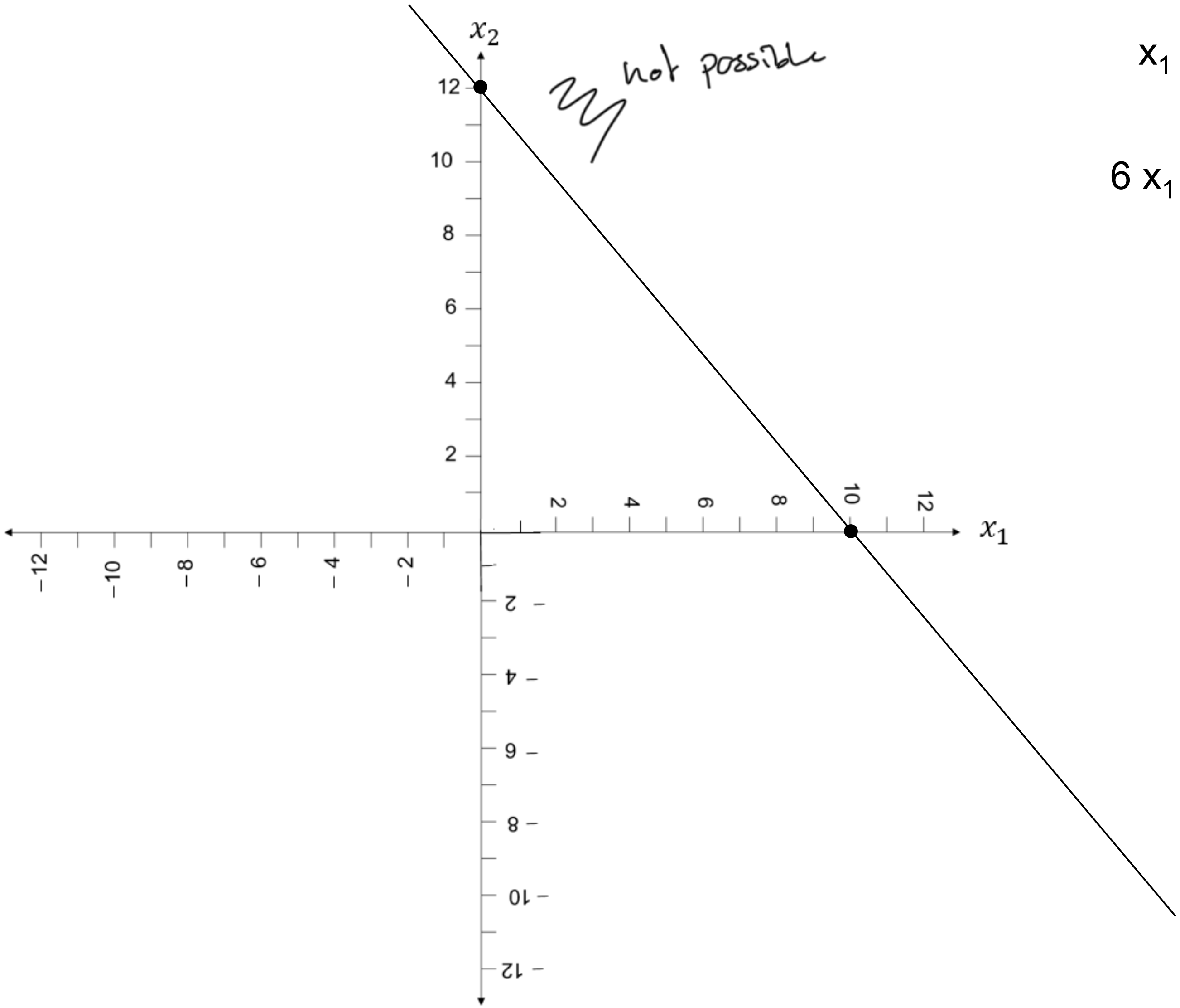
in n-space

$\sum_{j=1}^{n} a_j \cdot x_j \cdots$



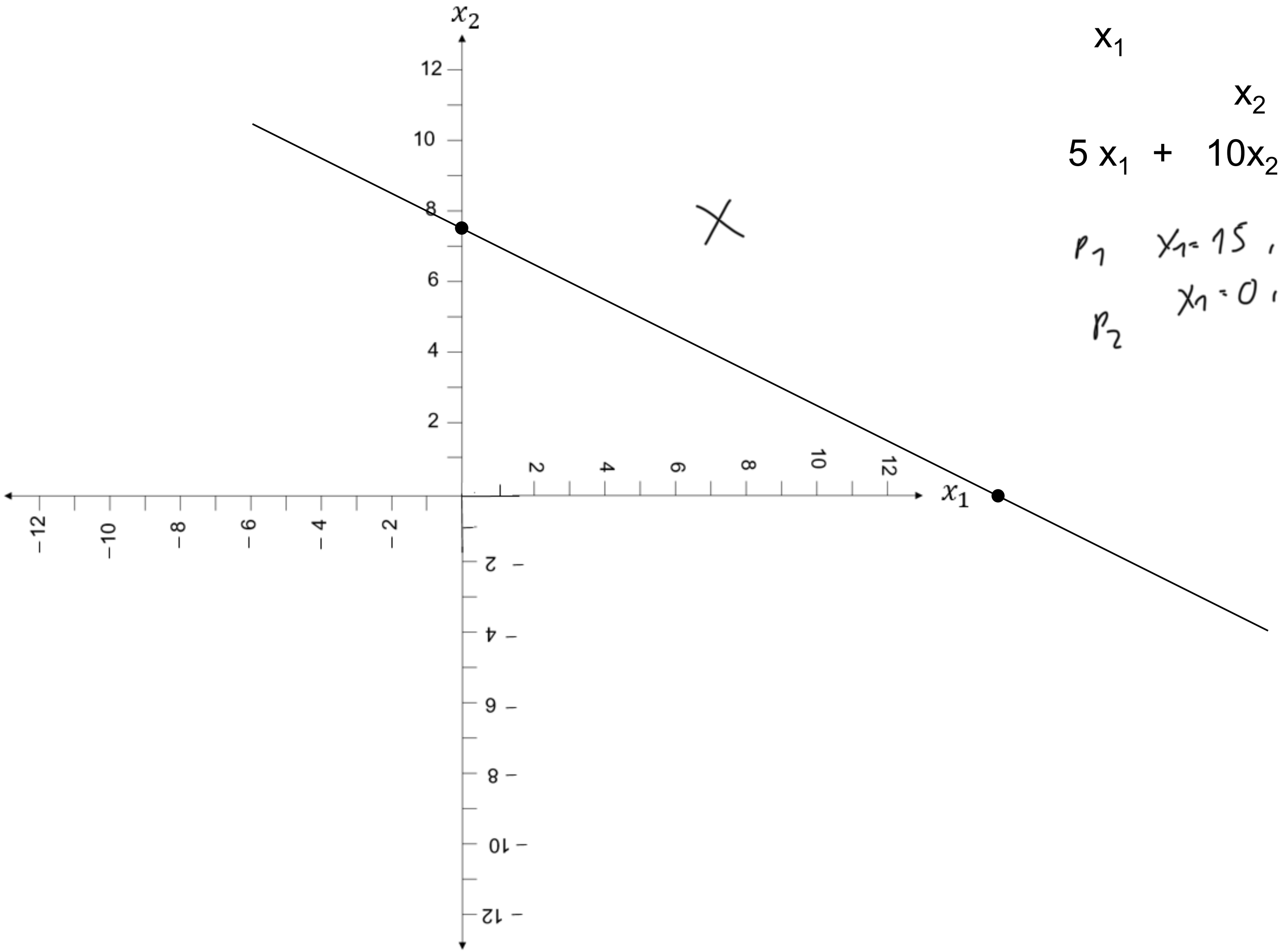
$$\begin{array}{ccccc} x_1 & & & \in & \mathbb{R} \\ & & x_2 & \in & \mathbb{R} \\ 6\,x_1 & + & 5\,x_2 & = & 60 \end{array}$$

Halfspace from the Production Constraint



$$
\begin{array}{rcrcl}
x_1 & & & \in & \mathbb{R} \\
 & & x_2 & \in & \mathbb{R} \\
6\,x_1 & + & 5\,x_2 & \leq & 60
\end{array}
$$

Halfspace from the Warehouse Constraint



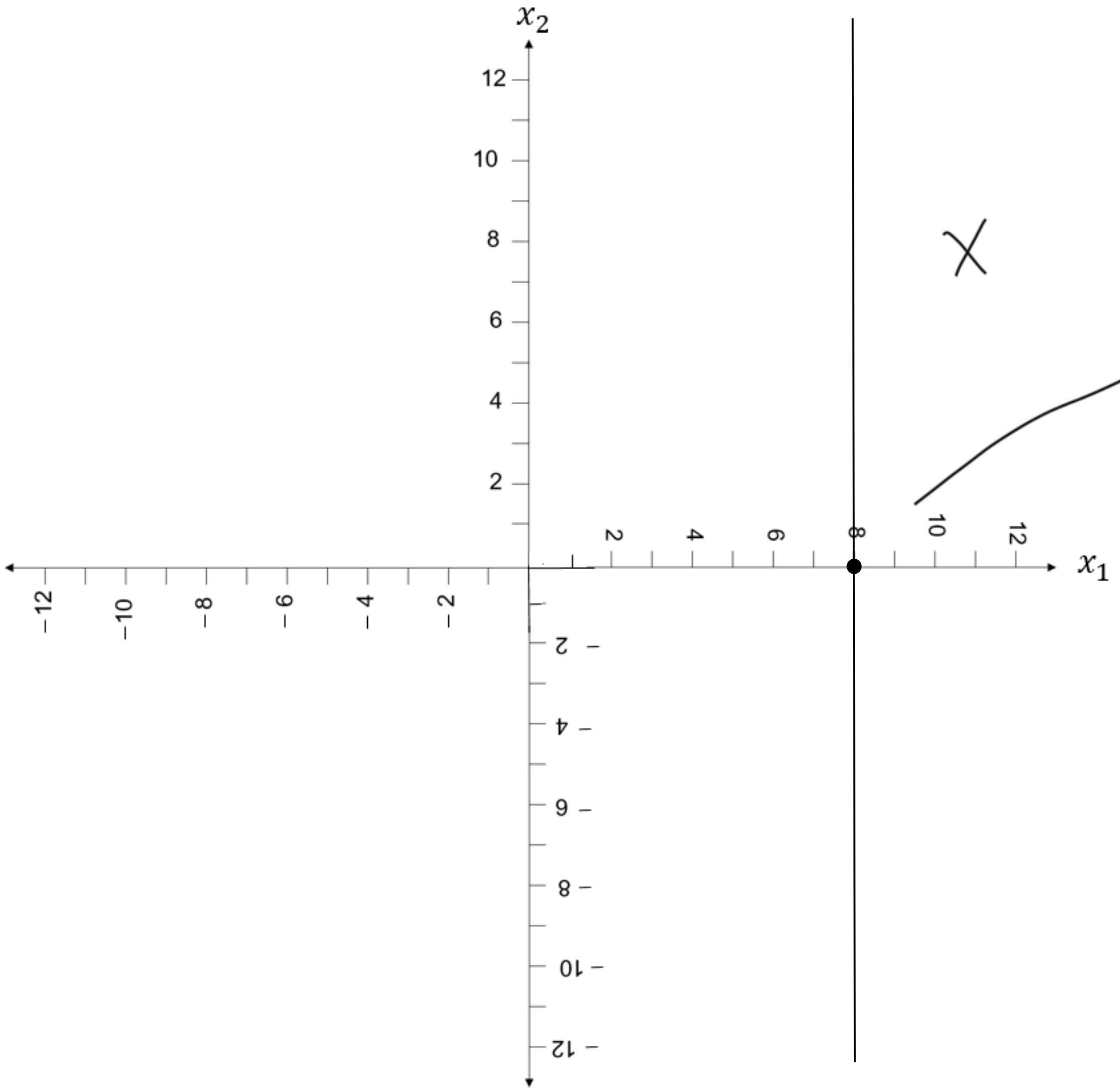
$$\begin{array}{rcl} x_1 & \in & \mathbb{R} \\ x_2 & \in & \mathbb{R} \\ 5x_1 + 10x_2 & \leq & 75 \end{array}$$

p_1 $x_1 = 15$, $x_2 = 0$

$x_1 = 0$, $x_2 = 7.5$

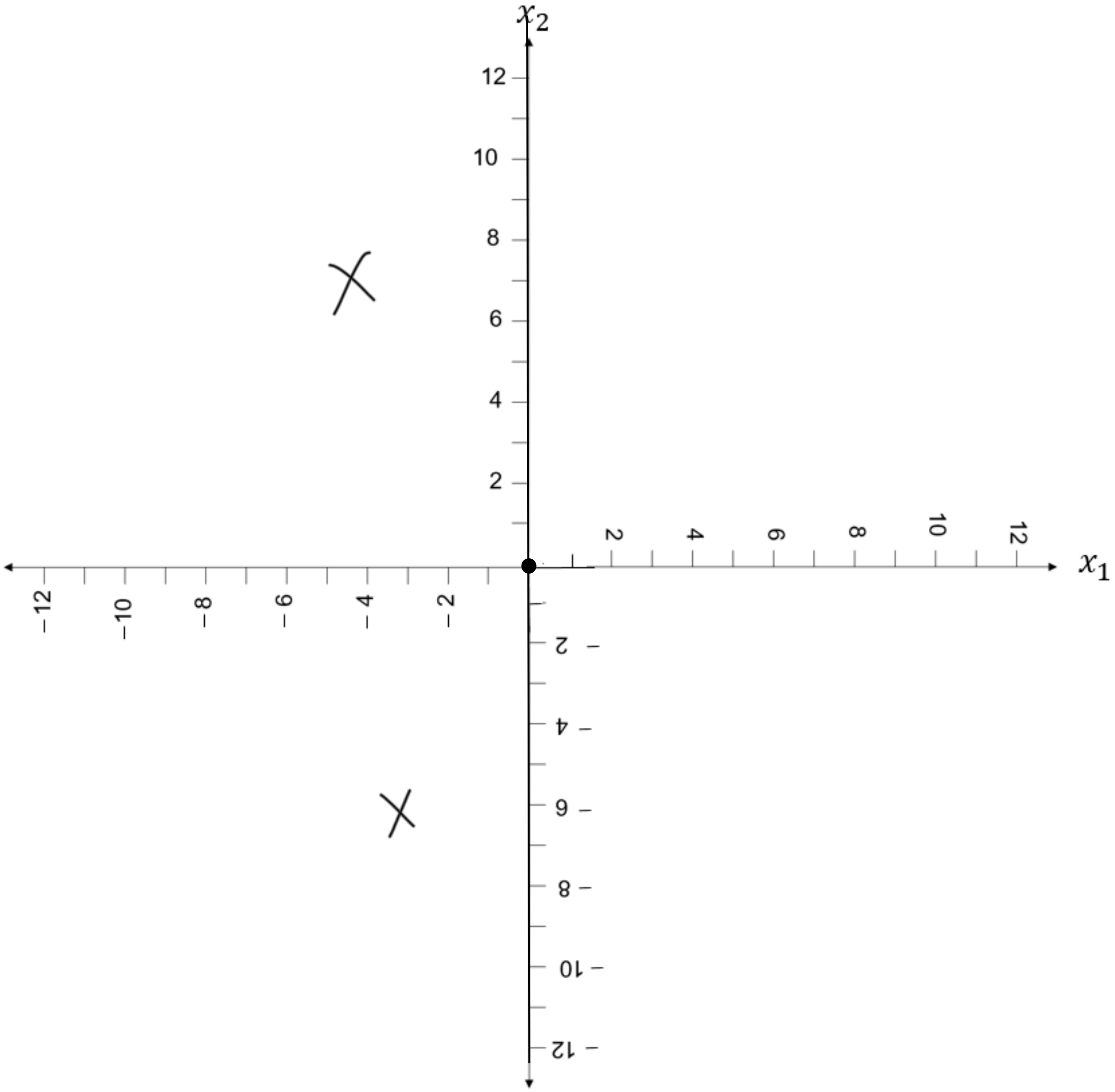
p_2

Halfspace from the Market Constraint



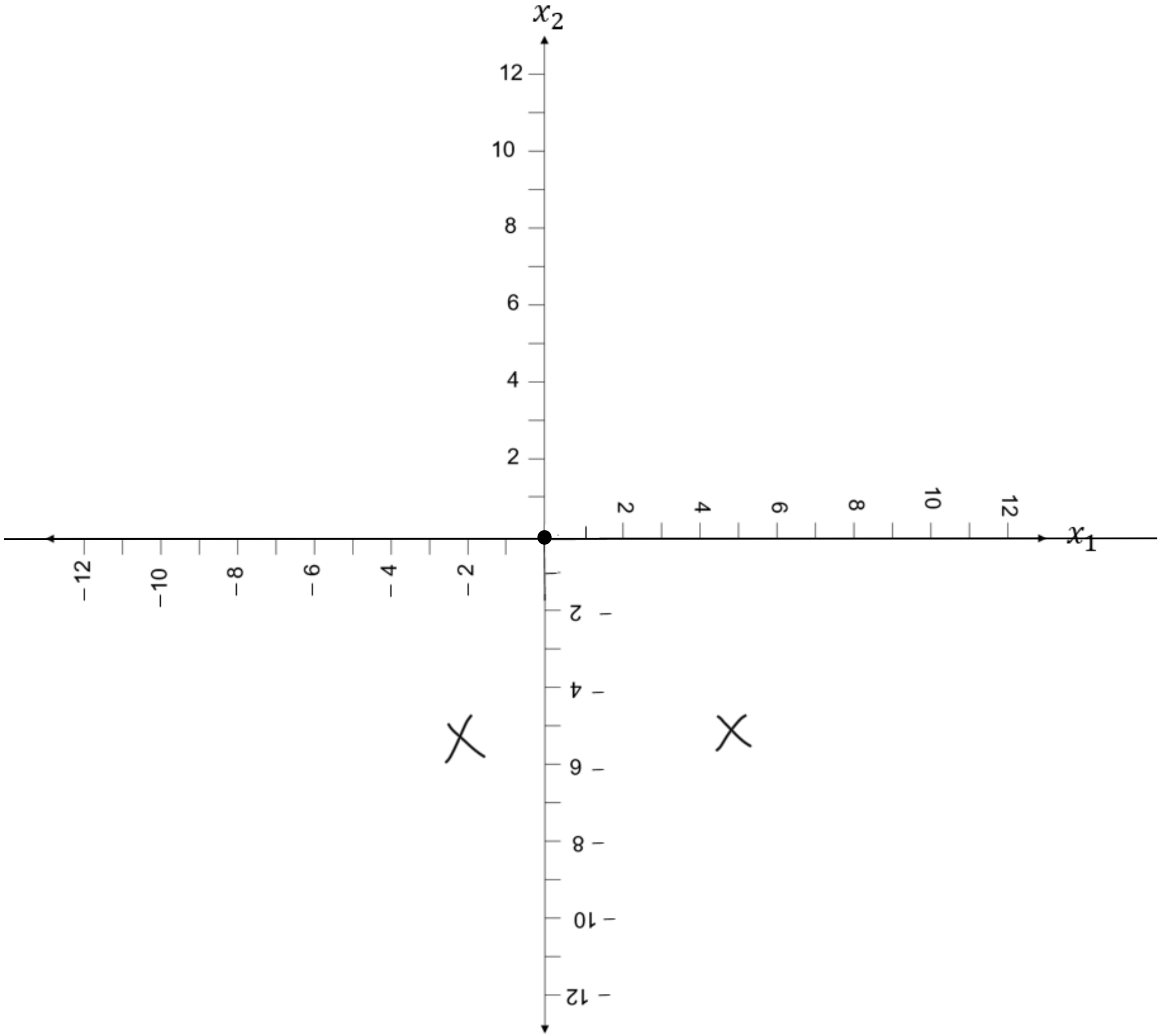
$$\begin{array}{lll} x_1 & & \in \mathbb{R} \\ & x_2 & \in \mathbb{R} \\ 1\,x_1 & & \leq 8 \end{array}$$

Halfspace from the Non-Negativity Constraint of x_1



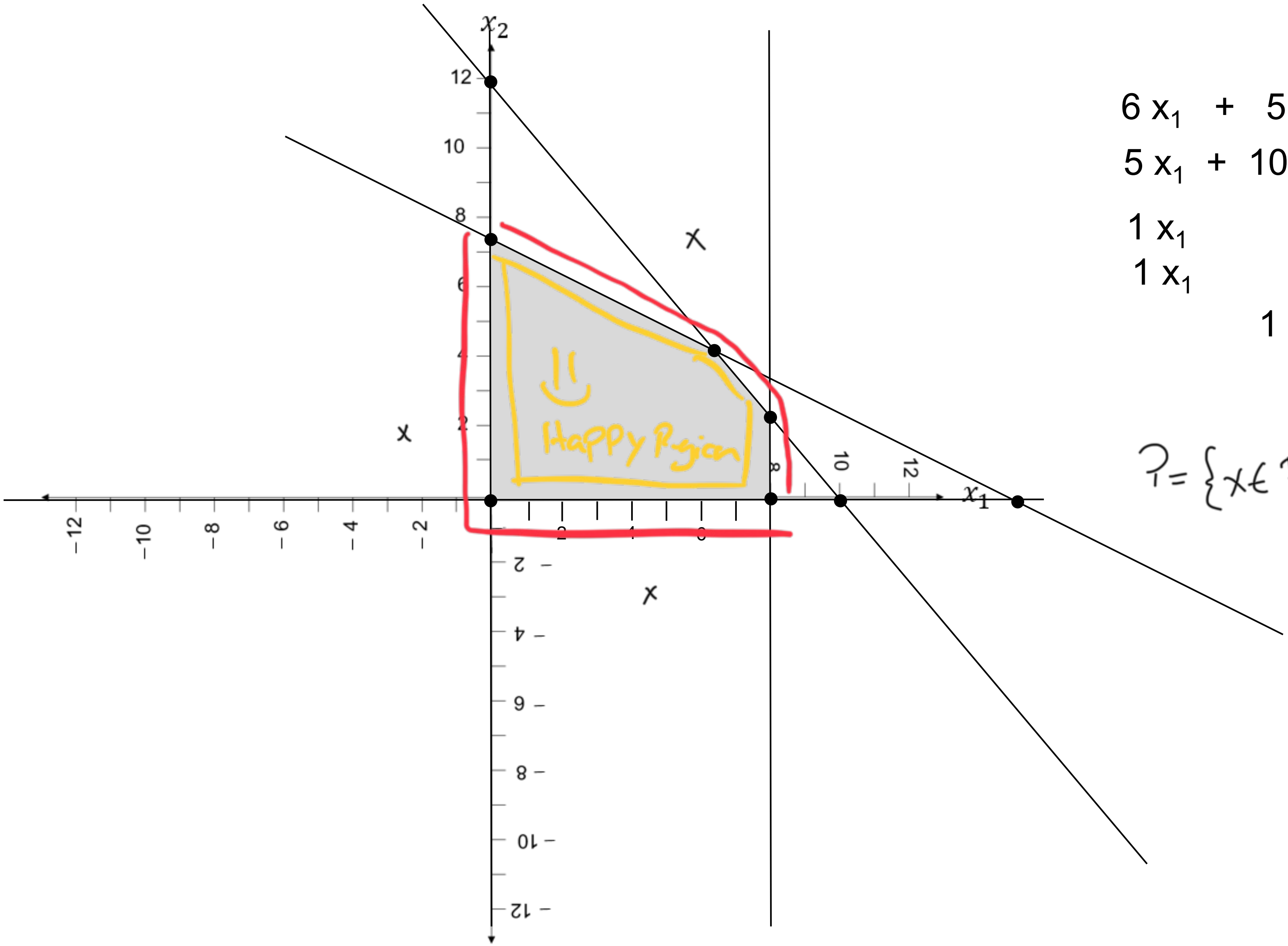
x_1		$\in$	$\mathbb{R}$
	x_2	$\in$	$\mathbb{R}$
$1\ x_1$		$\geq$	0

Halfspace from the Non-Negativity Constraint of x_2



$$
\begin{array}{lcc}
x_1 & \in & \mathbb{R} \\
x_2 & \in & \mathbb{R} \\
1\ x_2 & \geq & 0
\end{array}
$$

Polyhedron of the Beer Glass LP

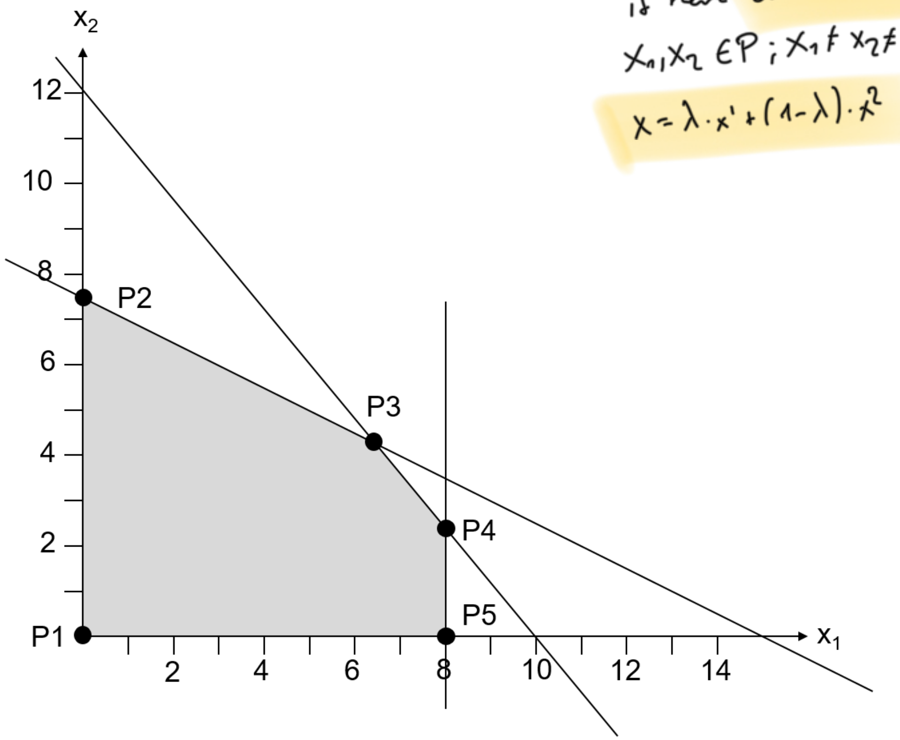


$$\begin{array}{rcrcr}
6\,x_1 & + & 5\,x_2 & \leq & 60 \\
5\,x_1 & + & 10\,x_2 & \leq & 75 \\
1\,x_1 & & & \leq & 8 \\
1\,x_1 & & & \geq & 0 \\
 & & 1\,x_2 & \geq & 0
\end{array}$$

$P = \{x \in \mathbb{R} \mid A \cdot x \leq b\}$

Extreme Points

Given polyhedron $P \in \mathbb{R}^n$
Vector $x \in P$ is extreme point of P
if there are no two vectors
$x_1, x_2 \in P$; $x_1 \neq x_2 \neq x$; $\lambda \in [0,1]$ such that
$x = \lambda \cdot x^1 + (1-\lambda) \cdot x^2$

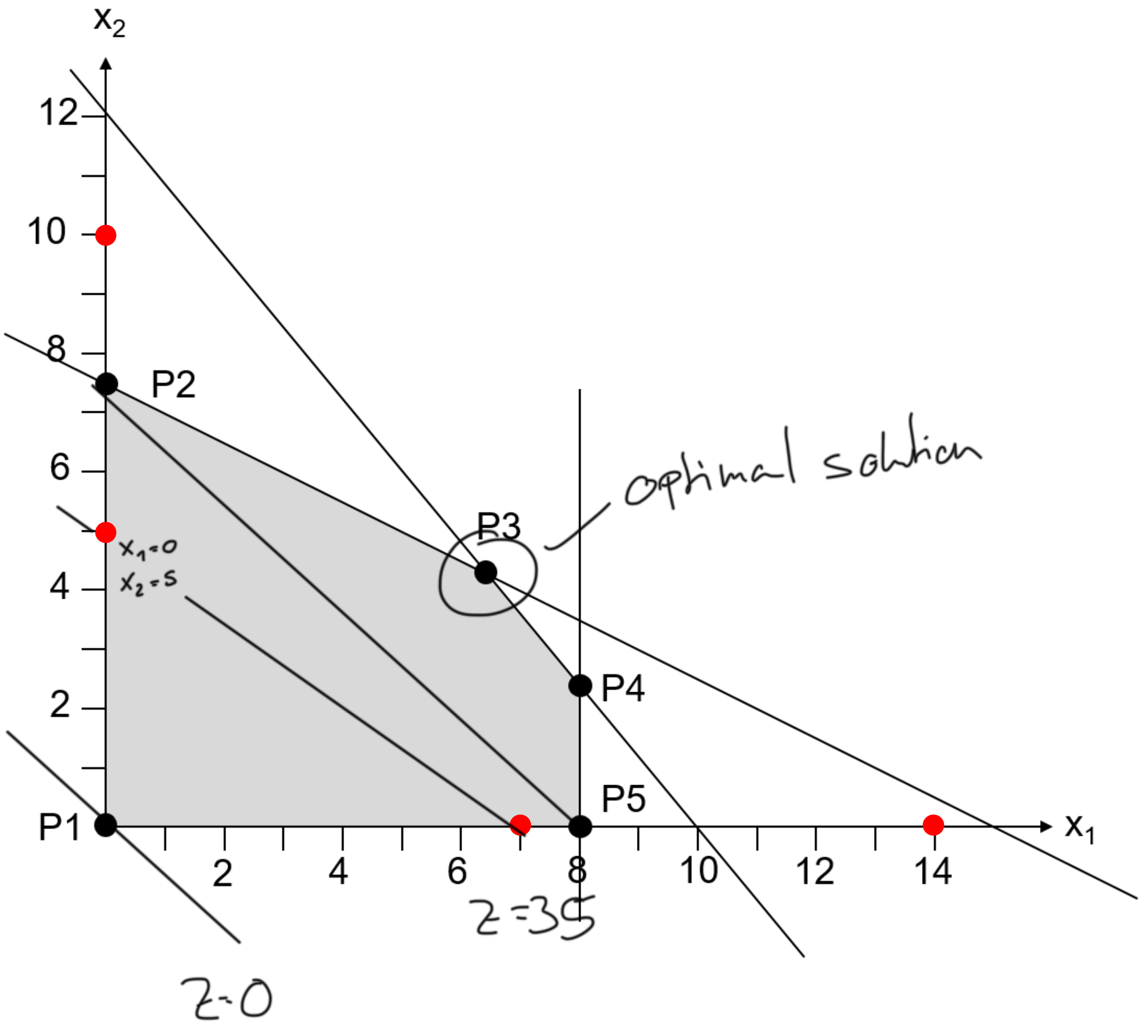


Objective Function

Maximize z = $5x_1 + 7x_2$

$x_1 = 0; x_2 = 5 \Rightarrow z = 35$

$x_1 = 7; x_2 = 0 \Rightarrow z = 35$



Determining the Optimal Solution

Demand constraint ignored?

Line production constraint: $6\,x_1 + 5\,x_2 = 60$

Line warehouse constraint: $5\,x_1 + 10\,x_2 = 75$

Gauss algorithm

	x_1	x_2	b
(1)	6	5	60
(2)	5	10	75
(3)	1	0.83	10
(4)	0	5.83	25
(5)	1	0	6.43
(6)	0	1	4.29

(3) = (1) / 6

$(4) = \left((2) - \frac{5}{6}\cdot(1)\right)/5$ hmm

$$A \cdot x = b$$

$$\begin{pmatrix} 6 & 5 \\ 5 & 10 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 60 \\ 75 \end{pmatrix}$$

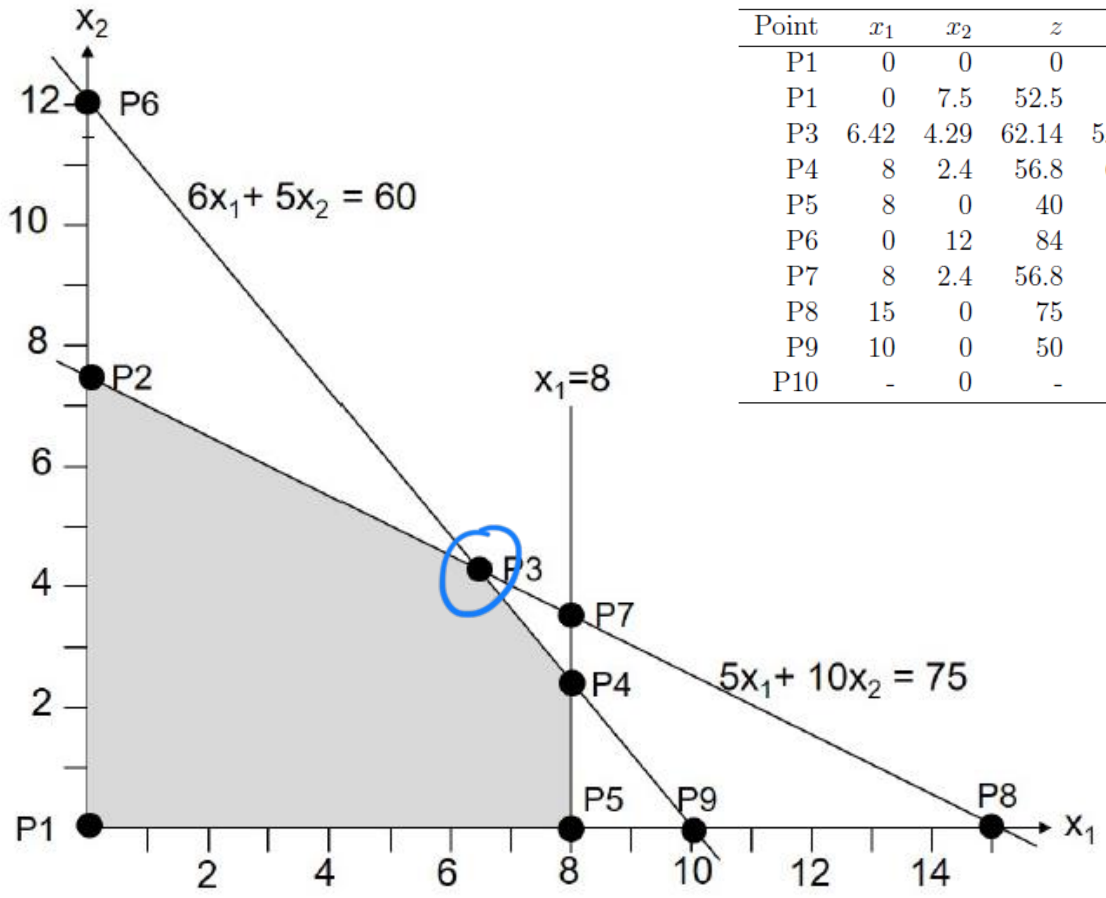
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \overline{b_1} \\ \overline{b_2} \end{pmatrix}$$

$x_1 = 6.43;\ x_2 = 4.29$

Determining all Extreme Points

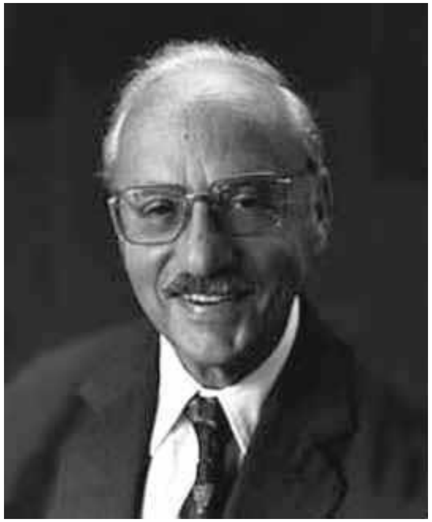
- An extreme point in our example with two variables is defined by the intersection of two lines.
- An extreme point in an LP with n variables is defined by the intersection of n of the $m > n$ hyperplanes (one for each constraint including non-negativity constraints).
- If we have m hyperplanes and $n \leq m$ variables, the number of possible extreme points is:
- For the beer glass LP, we have $m = 5$ and $n = 2$, which gives

Extreme Points for the Beer Glass LP



Point	x_1	x_2	z	Intersection of lines		Feasible	Note
P1	0	0	0	$x_1 = 0$	$x_2 = 0$	Yes	
P1	0	7.5	52.5	$x_1 = 0$	$5x_1 + 10x_2 = 75$	Yes	
P3	6.42	4.29	62.14	$5x_1 + 10x_2 = 75$	$6x_1 + 5x_2 = 60$	Yes	Optimal
P4	8	2.4	56.8	$6x_1 + 5x_2 = 60$	$x_1 = 8$	Yes	
P5	8	0	40	$x_1 = 8$	$x_2 = 0$	Yes	
P6	0	12	84	$x_1 = 0$	$6x_1 + 5x_2 = 60$	No	
P7	8	2.4	56.8	$x_1 = 8$	$5x_1 + 10x_2 = 75$	No	
P8	15	0	75	$x_2 = 0$	$5x_1 + 10x_2 = 75$	No	
P9	10	0	50	$x_2 = 0$	$6x_1 + 5x_2 = 60$	No	
P10	-	0	-	$x_1 = 0$	$x_1 = 8$	No	Not existing

3.2 Simplex Algorithm



Developed by Georg Dantzig in 1947

Professor for Operations Research and Computer Science
at Stanford University, USA

(* 8.11.1914 in Portland, USA, † 13.5.2005 in Stanford, USA)

Moves between neighbors (intersects) until z no longer improves → Optimization problem ☺

Standard Form of the LP

Transforming the non-linear Program into a linear one by adding factors

$$\begin{array}{lcccccccc}
 & & & & x_3 & x_4 & x_5 & & \\
\text{Maximize } z = & 5\,x_1 & + & 7\,x_2 & 0 & & & & \\
\text{subject to} & & & & & & & & \\
 & 6\,x_1 & + & 5\,x_2 & 1 & 0 & 0 & \cancel{\leq} = & 60 \\
 & 5\,x_1 & + & 10x_2 & 0 & 1 & 0 & \cancel{\leq} = & 75 \\
 & 1\,x_1 & & & 0 & 0 & 1 & \cancel{\leq} = & 8 \\
\text{non-neg. constr.} & 1\,x_1 & & & & & & \geq & 0 \\
 & & & 1\,x_2 & & & & \geq & 0 \\
 & & & & 1\,x_3 & 0\,x_4 & 0\,x_5 & \geq & 0 \\
 & & & & 0\,x_3 & 1\,x_4 & 0\,x_5 & \geq & 0 \\
 & & & & 0\,x_3 & 0\,x_4 & 1\,x_5 & \geq & 0
\end{array}$$

1. All variables are nonnegative.
2. Constraints, except non-negativity constraints of variables, are stated as equalities.
3. The right-hand-side of each constraint is nonnegative.
4. For each constraint, except the non-negativity constraints we have a variable with coefficient 1 in that constraint and coefficients 0 in all other constraints and in the objective function.

Note: The standard form is also called cannonical form

Decision Variables and Slack Variables

→ How much to produce → Measure of unused capacity

Maximize z = $5x_1 + 7x_2 + 0x_3 + 0x_4 + 0x_5$

subject to

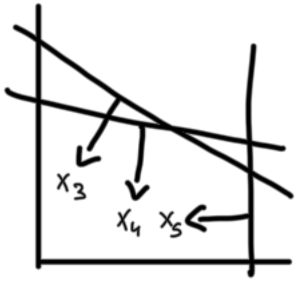
$$\begin{bmatrix} 6x_1 + 5x_2 \\ 5x_1 + 10x_2 \\ 1x_1 + 0x_2 \end{bmatrix} + \overset{A'}{\begin{bmatrix} 1x_3 + 0x_4 + 0x_5 \\ 0x_3 + 1x_4 + 0x_5 \\ 0x_3 + 0x_4 + 1x_5 \end{bmatrix}} = \overset{B'}{\begin{bmatrix} 60 \\ 75 \\ 8 \end{bmatrix}} \Rightarrow \begin{matrix} x_3 = 60 \\ x_4 = 75 \\ x_5 = 8 \end{matrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1x_1 \geq 0$$

$$1x_2 \geq 0$$

$$A' \cdot x = B'$$



Basic Variables, Non-Basic Variables, and Basis

↪ not set to 0,

⟶ $x_1=0; x_2=0$ → Define hyperplanes, but not part of the solution

$$\text{Maximize } z = \overbrace{5x_1 + 7x_2}^{\text{Decision}} + \overbrace{0x_3 + 0x_4 + 0x_5}^{\text{Slack}}$$

subject to

$$\begin{array}{rcrcrcrcrcl}
6x_1 & + & 5x_2 & + & 1x_3 & + & 0x_4 & + & 0x_5 & = & 60 \\
5x_1 & + & 10x_2 & + & 0x_3 & + & 1x_4 & + & 0x_5 & = & 75 \\
1x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & + & 1x_5 & = & 8 \\
1x_1 & & & & & & & & & \geq & 0 \\
 & & 1x_2 & & & & & & & \geq & 0
\end{array}$$

You call a variable a slack or surplus variable when it is introduced to convert an inequality constraint into an equality. This is a permanent designation based on the constraint type.

You call a variable basic or non-basic based on its status at a specific corner point (vertex) during the Simplex algorithm's iterative solution process. This designation changes as the algorithm moves from one corner point to the next.

Basis: Set of basic vars $[x_3, x_4, x_5]$

^T = Transposed

Matrix Notation of the LP

$$\text{Maximize } z = \left[5x_1 + 7x_2 + 0x_3 + 0x_4 + 0x_5\right]^{c^T}$$

subject to

$$A\begin{bmatrix} 6x_1 + 5x_2 + 1x_3 + 0x_4 + 0x_5 \\ 5x_1 + 10x_2 + 0x_3 + 1x_4 + 0x_5 \\ 1x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} 60 \\ 75 \\ 8 \end{bmatrix} b \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$1x_1 \geq 0$$

$$1x_2 \geq 0$$

Objective: $\max\ c^T \cdot x$

s.t. $A \cdot x = b$

$(x \geq 0)$

$$\rightarrow \max\ (5\ 7\ 0\ 0\ 0) \cdot \overset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}} \begin{matrix} = 0 \\ = 0 \\ \\ \\ \\ \end{matrix}$$

$$\text{s.t. } \begin{pmatrix} 6 & 5 & 1 & 0 & 0 \\ 5 & 10 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 60 \\ 75 \\ 8 \end{pmatrix}$$

Tableau Notation of the LP (also standard form)

$5 \cdot 0 + 7 \cdot 0 + 0 \cdot 0.6 + 0 \cdot 0.75 + 0 \cdot 0.08 = 0$

$$\text{Maximize } z = 5x_1 + 7x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$
\begin{aligned}
6x_1 + 5x_2 + 1x_3 + 0x_4 + 0x_5 &= 60\\
5x_1 + 10x_2 + 0x_3 + 1x_4 + 0x_5 &= 75\\
1x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 &= 8\\
1x_1 &\geq 0\\
1x_2 &\geq 0
\end{aligned}
$$

BV	Value	z	x_1	x_2	x_3	x_4	x_5
x_3	60	0	6	5	1	0	0
x_4	75	0	5	10	0	1	0
x_5	8	0	1	0	0	0	1
$-z$	0	-1	5	7	0	0	0

$$0 = -z + 5x_1 + 7x_2 + 0x_3 + 0x_4 + 0x_5$$

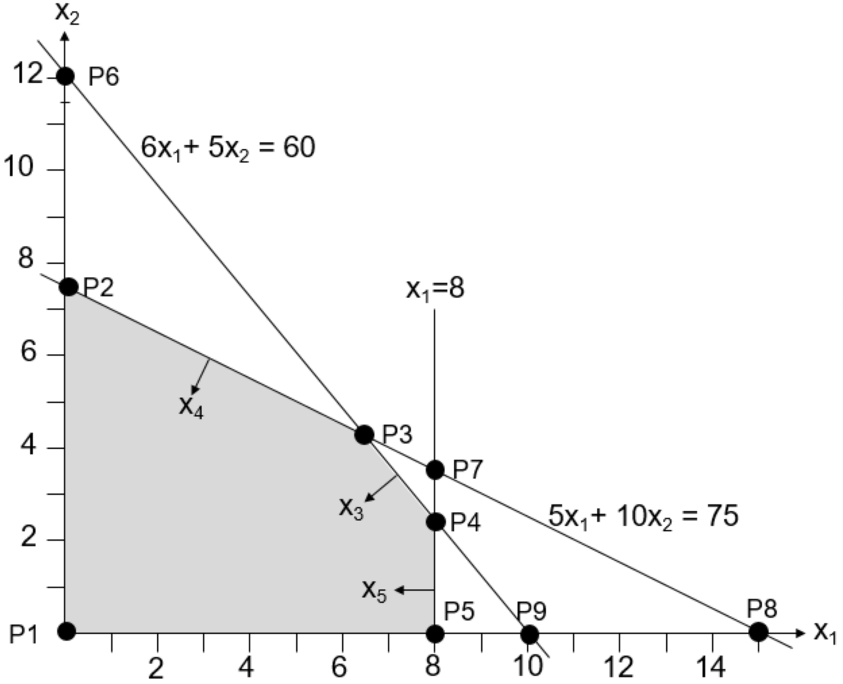
Reformed objective func. to adhere to std. form

Neighbor Basis

Definition 12 *Given a basis with m basic variables and $n - m$ non-basic variables, an* **adjacent basis (or neighbor basis)** *is a basis where exactly one former non-basic variable becomes a basic variable and one former basic variable becomes a non-basic variable.*

OFV = Objective Function Value

Finding the Neighbor Basis and Tableau Transformation



BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	60	6	5	1	0	0	1)
x_4	75	5	$\circled{10}$ Pivot el. in 2nd	0	1	0	2)
x_5	8	1	0	0	0	1	3)
-z	0	5	7	0	0	0	4)

↑ alwas the large NBV

BV: { x_2

1st solution: P_1 BV : $x_3 = 60$; $x_4 = 75$; $x_5 = 8$

NBV : $x_1 = 0$; $x_2 = 0$

OFV : $z = 0$

2nd : Move along longer NBV & remove pivot (?)

P_2 BV: $\{x_2, x_3, x_5\}$

NBV: $\{x_1, x_4\}$

OFV: 52.5

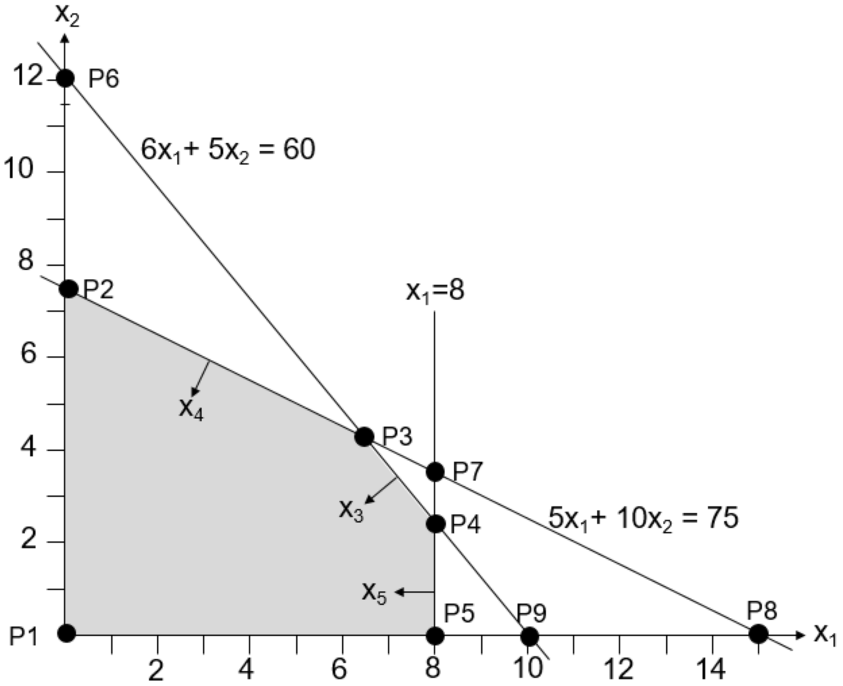
BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	22.5	3.5	0	1	-0.5	0	5) = 1) - 5 · 6)
x_2	7.5	0.5	①	0	0.1	0	6) = $\frac{2)}{10}$
x_5	8	1	0	0	0	1	7)
-z	-52.5	1.5	0	0	-0.7	0	8) = 4) - 7 · 6)

↓ 2nd step / 3rd point

Pivot Element

Definition 13 *The* **pivot element** (i, j) *of a tableau is determined by first selecting the non-basic variable* x_j *with largest positive coefficient value* c_j *and second by selecting the constraint* i *with* $a_{i,j} > 0$*, for which* $\frac{b_i}{a_{i,j}}$ *is smallest.*

Second Pivot Element and Tableau Transformation



BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	22.5	3.5	0	1	-0.5	0
x_2	7.5	0.5	1	0	0.1	0
x_5	8	1	0	0	0	1
-z	-52.5	1.5	0	0	-0.7	0

BV	Value	x_1	x_2	x_3	x_4	x_5	
x_1	6.42	1	0	0.28	-0.14	0	1) / 35
x_2	4.28	0	1	-0.14	0.17	0	2) -
x_5	1.57	0	0	-0.28	0.24	1	
-z	-62.14	0	0	-0.42	-0.48	0	

3.2.5

Formal Presentation of the Simplex Algorithm

Note: In this Chapter we always assume a maximization problem.

Simplex Algorithm

Prerequisite: LP with in standard form with max objective.

(1) If $\bar{c}_j \leq 0$ for j=1,…,n stop; Optimal solution.
Otherwise, continue only if there is at least one $\bar{c}_j > 0$.

Notation:

n = number of variables

j = variable index

$\bar{c}_j$ = objective function coefficient of the current tableau.

Simplex Algorithm

Prerequisite: LP in standard form with max objective.

(1) If $\bar{c}_j \leq 0$ for j=1,…,n stop; Optimal solution.
Otherwise, continue only if there is at least one $\bar{c}_j > 0$.

(2) **Choose pivot column s.**

$$\bar{c}_s = \max_{j=1}^{n} \{ \bar{c}_j \mid \bar{c}_j > 0 \}$$

If $\bar{a}_{i,s} \leq 0$ für i=1,…,m, stop;
Special case: Unbounded solution.
Continue only if there is at least one $\bar{a}_{i,s} > 0$.

Notation:

n = number of variables

j = variable index

$\bar{c}_j$ = objective function coefficient of the current tableau.

m = number of constraints

i = constraint index

$\bar{a}_{i,s}$ = coefficient of element (i,s) in the current tableau

Simplex Algorithm

Prerequisite: LP in standard form with max objective.

(1) If $\bar{c}_j \leq 0$ for j=1,…,n stop; Optimal solution.
Otherwise, continue only if there is at least one $\bar{c}_j > 0$.

(2) **Choose pivot column s.**

$$\bar{c}_s = \max_{j=1}^{n} \{ \bar{c}_j \mid \bar{c}_j > 0 \}$$

If $\bar{a}_{i,s} \leq 0$ für i=1,…,m, stop;
Special case: Unbounded solution.
Continue only if there is at least one $\bar{a}_{i,s} > 0$.

(3) **Choose pivot row r**

$$\frac{\bar{b}_r}{\bar{a}_{r,s}} = \min_{i=1}^{m} \{ \frac{\bar{b}_i}{\bar{a}_{i,s}} \mid \bar{a}_{i,s} > 0 \}$$

Notation:

n = number of variables

j = variable index

$\bar{c}_j$ = objective function coefficient of the current tableau.

m = number of constraints

i = constraint index

$\bar{a}_{i,s}$= coefficient of element (i,s) in the current tableau

b_i= current right-hand-side

Simplex Algorithm

Prerequisite: LP in standard form with max objective.

(1) If $\bar{c}_j \leq 0$ for j=1,…,n stop; Optimal solution.
Otherwise, continue only if there is at least one $\bar{c}_j > 0$.

(2) **Choose pivot column s**

$$\bar{c}_s = \max_{j=1}^{n} \{ \bar{c}_j \mid \bar{c}_j > 0 \}$$

If $\bar{a}_{i,s} \leq 0$ für i=1,…,m, stop;
Special case: Unbounded solution.
Continue only if there is at least one $\bar{a}_{i,s} > 0$.

(3) **Choose pivot row r**

$$\frac{\bar{b}_r}{\bar{a}_{r,s}} = \min_{i=1}^{m} \{ \frac{\bar{b}_i}{\bar{a}_{i,s}} \mid \bar{a}_{i,s} > 0 \}$$

(4) **Change basic variables and re-establish canonical form**

Notation:

n = number of variables

j = variable index

$\bar{c}_j$ = objective function coefficient of the current tableau.

m = number of constraints

i = constraint index

$\bar{a}_{i,s}$= coefficient of element (i,s) in the current tableau

b_i= current right-hand-side

Simplex Algorithm

Prerequisite: LP in standard form with max objective.

(1) If $\bar{c}_j \leq 0$ for j=1,…,n stop; Optimal solution.
Otherwise, continue only if there is at least one $\bar{c}_j > 0$.

(2) **Choose pivot column s.**

$$\bar{c}_s = \max_{j=1}^{n} \{ \bar{c}_j \mid \bar{c}_j > 0 \}$$

If $\bar{a}_{i,s} \leq 0$ für i=1,…,m, stop;
Special case: Unbounded solution.
Continue only if there is at least one $\bar{a}_{i,s} > 0$.

(3) **Choose pivot row r**

$$\frac{\bar{b}_r}{\bar{a}_{r,s}} = \min_{i=1}^{m} \{ \frac{\bar{b}_i}{\bar{a}_{i,s}} \mid \bar{a}_{i,s} > 0 \}$$

(4) **Change basis variables and re-establish a canonical form**

(5) **Go to step (1)**

Notation:

n = number of variables

j = variable index

$\bar{c}_j$ = objective function coefficient of the current tableau.

m = number of constraints

i = constraint index

$\bar{a}_{i,s}$= coefficient of element (i,s) in the current tableau

b_i= current right-hand-side

Simplex Algorithm

Prerequisite: LP in standard form with max objective.

(1) If $\bar{c}_j \leq 0$ for j=1,…,n stop; Optimal solution.
Otherwise, continue only if there is at least one $\bar{c}_j > 0$.

(2) **Choose pivot column s.**

$$\bar{c}_s = \max_{j=1}^{n} \{ \bar{c}_j \mid \bar{c}_j > 0 \}$$

If $\bar{a}_{i,s} \leq 0$ für i=1,…,m, stop;
Special case: Unbounded solution.
Continue only if there is at least one $\bar{a}_{i,s} > 0$.

(3) **Choose pivot row r**

$$\frac{\bar{b}_r}{\bar{a}_{r,s}} = \min_{i=1}^{m} \{ \frac{\bar{b}_i}{\bar{a}_{i,s}} \mid \bar{a}_{i,s} > 0 \}$$

(4) **Change basis variables and re-establish canonical form**

(5) **Go to step (1)**

Property 5: The Simplex ends with one of the two results:

1. an optimal solution or
2. an unbounded solution.

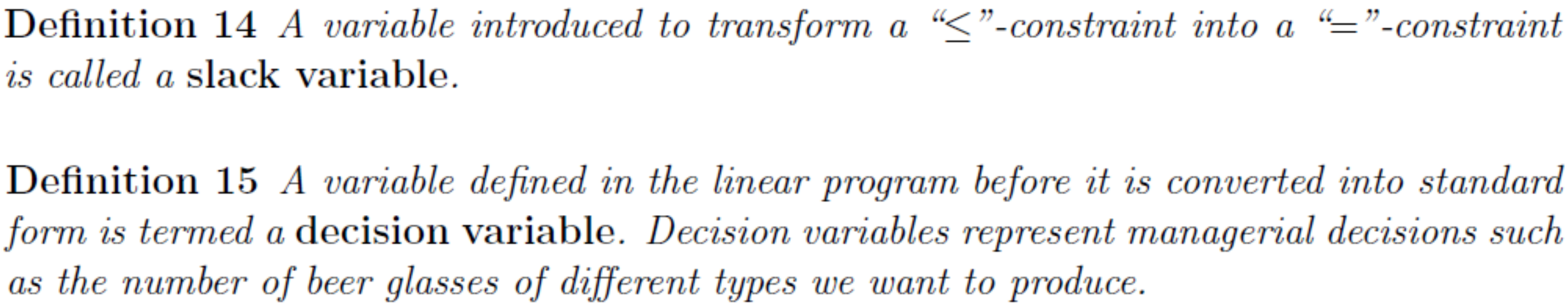
3.3

Deriving the Standard Form

Note: In this Chapter we always assume a maximization problem.

Less-Than-Or-Equal Constraints

$40\,x_1 + 10\,x_2 + 6\,x_3 \qquad \leq 55$



Definition 14 *A variable introduced to transform a "$\leq$"-constraint into a "$=$"-constraint is called a* **slack variable***.*

Definition 15 *A variable defined in the linear program before it is converted into standard form is termed a* **decision variable***. Decision variables represent managerial decisions such as the number of beer glasses of different types we want to produce.*

Greater-Than-Or-Equal Constraints

$40\ x_1 + 10\ x_2 + 6\ x_3 \qquad \geq 32$

$40x_1 + 10x_2 + 6x_3 - 1x_4 + 1x_5 \qquad = 32$

$x_4 \qquad \geq 0$ surplus

$x_5 \qquad \geq 0$ artificial

Definition 16 *A variable introduced to transform a "$\geq$"-constraint into a "$=$"-constraint is called a* **surplus variable.**

Definition 17 *A variable introduced in "$=$"-constraint in order to have a basic variable for this constraint is called an* **artificial variable.**

Constraints with Negative Right-Hand-Side

$10 x_1 - 5 x_2 - 6 x_3 \leq - 30$ $\quad | \cdot (-1)$

$-10x_1 + 5x_x + 6x_3 - 1x_4 + 1x_5 = 30$

$x_4 \geqslant 0$ surplus

$x_5 \geqslant 0$ artificial

Equality Constraints

$5\ x_1 + 3\ x_2 = 20$

$5x_1 + 3x_2 + 1x_3 = 20$

$x_3 \geq 0$ artificial

Real-Valued Variables

Production - Change in Inventory = Demand

$x - \Delta L = 50$

$x \geq 0, \quad \Delta L \in \mathbf{R}$

Three cases

Increase
in inventory

$x > 50$

$\Downarrow$

$\Delta L > 0$

No change
in inventory

$x = 50$

$\Downarrow$

$\Delta L = 0$

Decrease
in Inventory

$x < 50$

$\Downarrow$

$\Delta L < 0$

Real-Valued Variables

$$x - \Delta L = 50$$

$$x \geq 0$$

$$\Delta L \in \mathbb{R}$$

3.3.6

Example for Transformation to Standard Form

Example for Transformation to Standard Form

Min $-1\ x_1 + 1\ x_2$

subject to

$$
\begin{aligned}
2\ x_1 + 1\ x_2 &\leq 10 \\
1\ x_1 + 2\ x_2 &\geq 3 \\
1\ x_1 + 1\ x_2 &= -5 \\
x_1 &\in \mathbf{R} \\
x_2 &\geq 0
\end{aligned}
$$

Transformation of Objective Function (1)

$\text{Min } -1\,x_1 + 1\,x_2$ (1)

subject to

$2\,x_1 + 1\,x_2 \leq 10$ (2)

$1\,x_1 + 2\,x_2 \geq 3$ (3)

$1\,x_1 + 1\,x_2 = -5$ (4)

$x_1 \in \mathbf{R}$ (5)

$x_2 \geq 0$ (6)

(7)

subject to

$2\,x_1 + 1\,x_2 \leq 10$ (2)

$1\,x_1 + 2\,x_2 \geq 3$ (3)

$1\,x_1 + 1\,x_2 = -5$ (4)

$x_1 \in \mathbf{R}$ (5)

$x_2 \geq 0$ (6)

Transformation of Constraint (2)

Max $1 x_1 - 1 x_2$ (1)

subject to

$2 x_1 + 1 x_2 \leq 10$ (2)

$1 x_1 + 2 x_2 \geq 3$ (3)

$1 x_1 + 1 x_2 = -5$ (4)

$x_1 \in \mathbf{R}$ (5)

$x_2 \geq 0$ (6)

Max $1 x_1 - 1 x_2$ (1)

subject to

(2)

Transformation of Constraint (3)

$$\text{Max } 1\,x_1 - 1\,x_2 \qquad (1)$$

subject to

$$2\,x_1 + 1\,x_2 + 1\,x_3 = 10 \qquad (2)$$

$$1\,x_1 + 2\,x_2 \geq 3 \qquad (3)$$

$$1\,x_1 + 1\,x_2 = -5 \qquad (4)$$

$$x_1 \in \mathbf{R} \qquad (5)$$

$$x_2 \geq 0 \qquad (6)$$

$$x_3 \geq 0 \qquad (7)$$

$$\text{Max } 1\,x_1 - 1\,x_2 \qquad (1)$$

subject to

$$2\,x_1 + 1\,x_2 + 1\,x_3 = 10 \qquad (2)$$

(3)

Transformation of Constraint (4)

$$
\begin{array}{lll}
\text{Max } 1\,x_1 - 1\,x_2 \qquad - M\,x_5 & & (1) \\
\text{subject to} & & \\
2\,x_1 + 1\,x_2 + 1\,x_3 & = 10 & (2) \\
1\,x_1 + 2\,x_2 \qquad - 1\,x_4 + 1\,x_5 & = 3 & (3) \\
1\,x_1 + 1\,x_2 & = -5 & (4) \\
x_1 & \in \mathbf{R} & (5) \\
x_2 & \geq 0 & (6) \\
x_3 & \geq 0 & (7) \\
x_4 & \geq 0 & (8) \\
x_5 & \geq 0 & (9)
\end{array}
$$

$$
\begin{array}{lll}
\text{Max } 1\,x_1 - 1\,x_2 \qquad - M\,x_5 & & (1) \\
\text{subject to} & & \\
2\,x_1 + 1\,x_2 + 1\,x_3 & = 10 & (2) \\
1\,x_1 + 2\,x_2 \qquad - 1\,x_4 + 1\,x_5 & = 3 & (3) \\
 & & (4) \\
x_1 & \in \mathbf{R} & (5) \\
x_2 & \geq 0 & (6) \\
x_3 & \geq 0 & (7) \\
x_4 & \geq 0 & (8) \\
x_5 & \geq 0 & (9)
\end{array}
$$

Transformation of Constraint (5)

$$1\,x_1 - 1\,x_2 \qquad - M\,x_5 - M\,x_6 \qquad (1)$$

subject to

$$\begin{array}{lr}
2\,x_1 + 1\,x_2 + 1\,x_3 \qquad = 10 & (2) \\
1\,x_1 + 2\,x_2 \qquad - 1\,x_4 + 1\,x_5 \qquad = \;\; 3 & (3) \\
-1\,x_1 - 1\,x_2 \qquad + 1\,x_6 \;=\; 5 & (4) \\
x_1 \qquad \in \mathbf{R} & (5) \\
x_2 \qquad \geq \;\; 0 & (6) \\
x_3 \qquad \geq \;\; 0 & (7) \\
x_4 \qquad \geq \;\; 0 & (8) \\
x_5 \qquad \geq \;\; 0 & (9) \\
x_6 \qquad \geq \;\; 0 & (10)
\end{array}$$

$$1\,x_1 - 1\,x_2 \qquad - M\,x_5 - M\,x_6 \qquad (1)$$

subject to

$$\begin{array}{lr}
2\,x_1 + 1\,x_2 + 1\,x_3 \qquad = 10 & (2) \\
1\,x_1 + 2\,x_2 \qquad - 1\,x_4 + 1\,x_5 \qquad = \;\; 3 & (3) \\
-1\,x_1 - 1\,x_2 \qquad + 1\,x_6 \;=\; 5 & (4) \\
 & (5) \\
x_2 \qquad \geq \;\; 0 & (6) \\
x_3 \qquad \geq \;\; 0 & (7) \\
x_4 \qquad \geq \;\; 0 & (8) \\
x_5 \qquad \geq \;\; 0 & (9) \\
x_6 \qquad \geq \;\; 0 & (10)
\end{array}$$

Transformation of Constraint (6)

$$\begin{array}{lr}
1\,x_1 - 1\,x_2 - M\,x_5 - M\,x_6 & (1) \\
\text{subject to} & \\
2\,x_1 + 1\,x_2 + 1\,x_3 = 10 & (2) \\
1\,x_1 + 2\,x_2 - 1\,x_4 + 1\,x_5 = 3 & (3) \\
-1\,x_1 - 1\,x_2 + 1\,x_6 = 5 & (4) \\
x_1 \in \mathbf{R} & (5) \\
x_2 \geq 0 & (6) \\
x_3 \geq 0 & (7) \\
x_4 \geq 0 & (8) \\
x_5 \geq 0 & (9) \\
x_6 \geq 0 & (10)
\end{array}$$

$$\begin{array}{lr}
1x^+_1 - 1x^-_1 - 1\,x_2 - M\,x_5 - M\,x_6 & (1) \\
\text{subject to} & \\
2x^+_1 - 2x^-_1 + 1\,x_2 + 1\,x_3 = 10 & (2) \\
1x^+_1 - 1x^-_1 + 2\,x_2 - 1\,x_4 + 1\,x_5 = 3 & (3) \\
-1x^+_1 + 1x^-_1 - 1\,x_2 + 1\,x_6 = 5 & (4) \\
x^+_1 \geq 0 & (11) \\
x^-_1 \geq 0 & (12) \\
x_2 \geq 0 & (6) \\
x_3 \geq 0 & (7) \\
x_4 \geq 0 & (8) \\
x_5 \geq 0 & (9) \\
x_6 \geq 0 & (10)
\end{array}$$

Result of the Transformation

$$\text{Min } -1\,x_1 + 1\,x_2 \qquad (1)$$

subject to

$$\begin{array}{lr}
2\,x_1 + 1\,x_2 \leq 10 & (2) \\
1\,x_1 + 2\,x_2 \geq \;\; 3 & (3) \\
1\,x_1 + 1\,x_2 = -5 & (4) \\
x_1 \in \mathbf{R} & (5) \\
x_2 \geq \;\; 0 & (6)
\end{array}$$

$$\text{Max } 1x^+_1 - 1x^-_1 - 1\,x_2 - M\,x_5 - M\,x_6 \qquad (1)$$

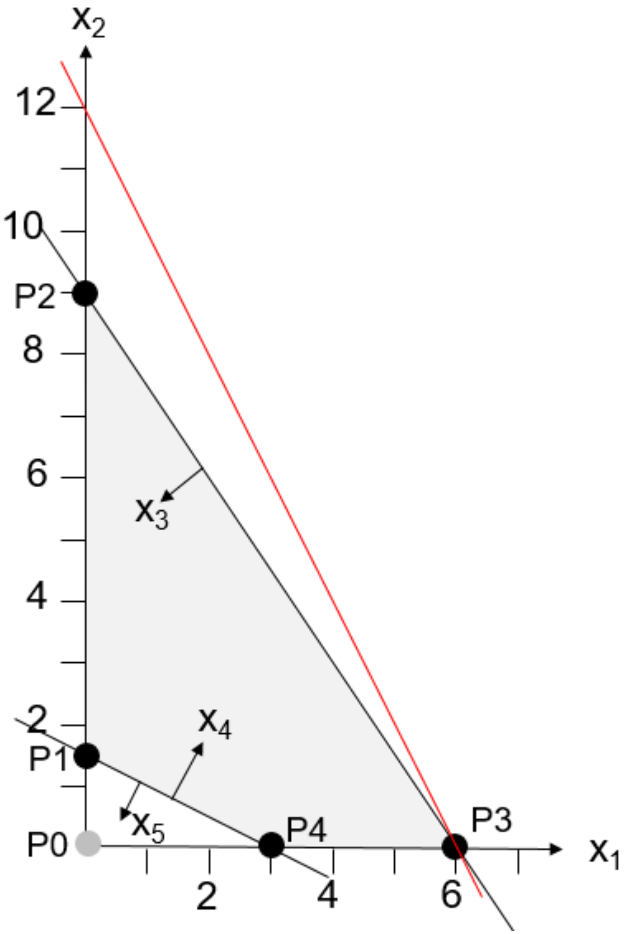
subject to

$$\begin{array}{lr}
2x^+_1 - 2x^-_1 + 1\,x_2 + 1\,x_3 = 10 & (2) \\
1x^+_1 - 1x^-_1 + 2\,x_2 - 1\,x_4 + 1\,x_5 = \;\; 3 & (3) \\
-1x^+_1 + 1x^-_1 - 1\,x_2 + 1\,x_6 = 5 & (4) \\
x^+_1 \geq 0 & (11) \\
x^-_1 \geq 0 & (12) \\
x_2 \geq 0 & (6) \\
x_3 \geq 0 & (7) \\
x_4 \geq 0 & (8) \\
x_5 \geq 0 & (9) \\
x_6 \geq 0 & (10)
\end{array}$$

3.4.1

Solving the LP with Artificial Variables: Two-Phase Method

Two-Phase Method: Standard Form and Objective Functions



Max $2\ x_1 + \ 1\ x_2$

subject to

$$3\ x_1 + 2\ x_2 \leq 18$$

$$1\ x_1 + 2\ x_2 \geq \ 3$$

$$x_1, x_2 \geq 0$$

Two-Phase Method: Standard Form and Objective Functions

$$\text{Max } w = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1x_5$$

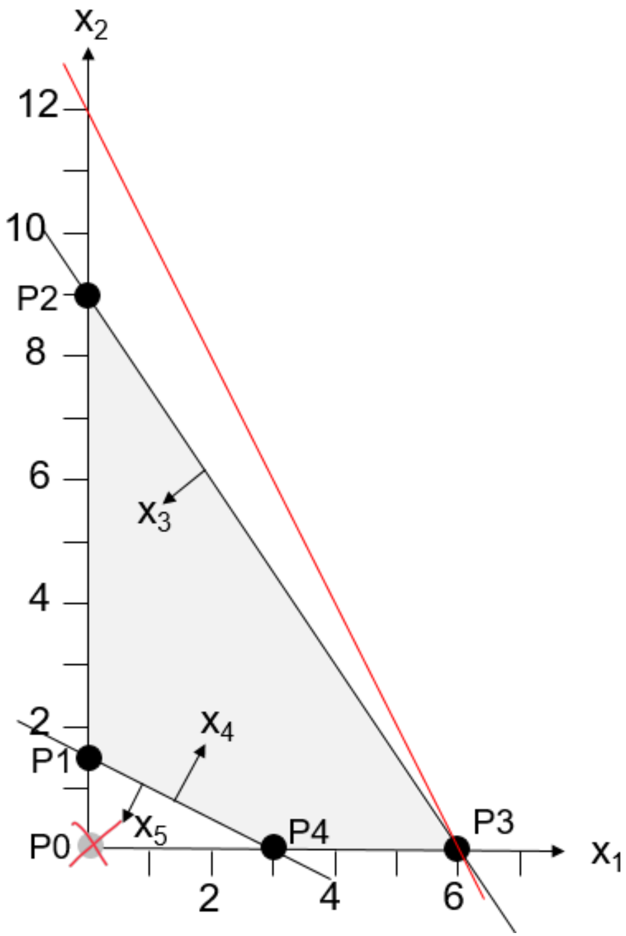
$$\text{Max } z = 2x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5$$

s.t.

$$\begin{aligned} 3x_1 + 2x_2 + 1x_3 + 0x_4 + 0x_5 &= 18 \\ 1x_1 + 2x_2 + 0x_3 - 1x_4 + 1x_5 &= 3 \\ x_i &\geq 0 \ (i = 1, \ldots, 5) \end{aligned}$$

BV	Value	x_1	x_2	x_3	x_4	x_5

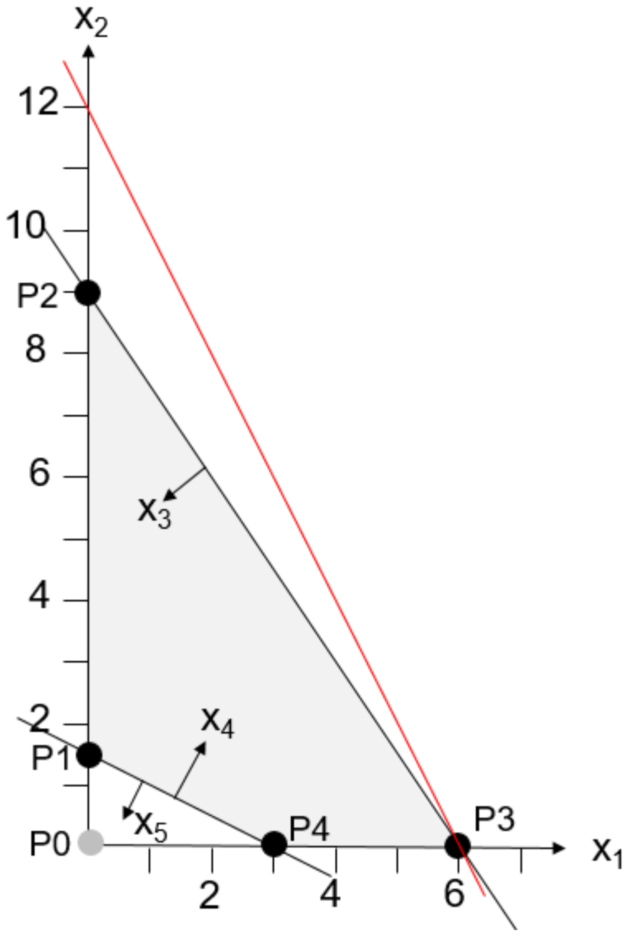
Two-Phase Method: Tableau with Objective Function *w*



BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	18	3	2	1	0	0	(1)
x_5	3	1	2	0	-1	1	(2)
$-w$	0	0	0	0	0	-1	(3)

BV	Value	x_1	x_2	x_3	x_4	x_5

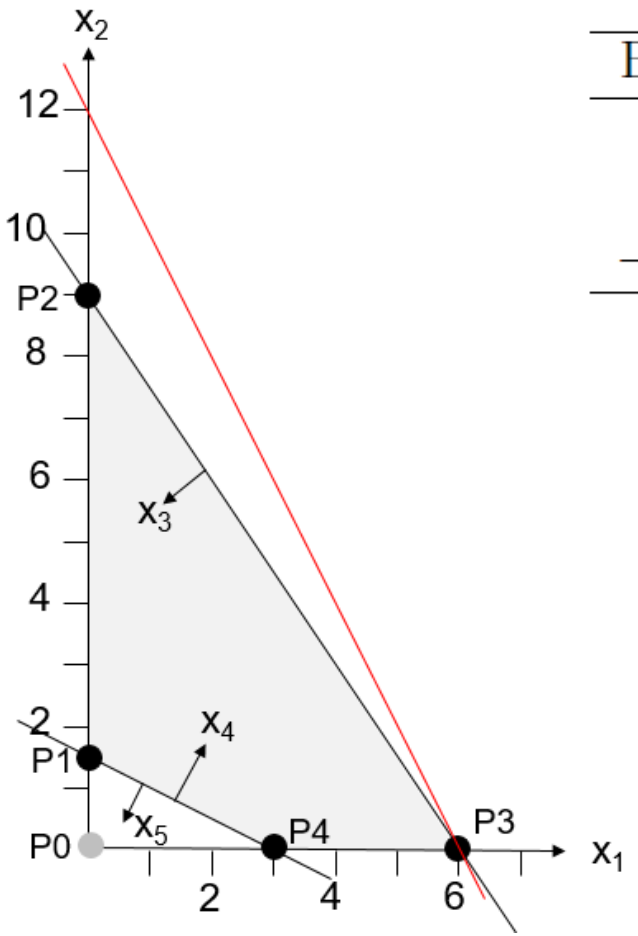
Two-Phase Method: Optimizing Objective Function *w*



BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	18	3	2	1	0	0	(4)
x_5	3	1	2	0	-1	1	(5)
$-w$	3	1	2	0	-1	0	(6)

BV	Value	x_1	x_2	x_3	x_4	x_5

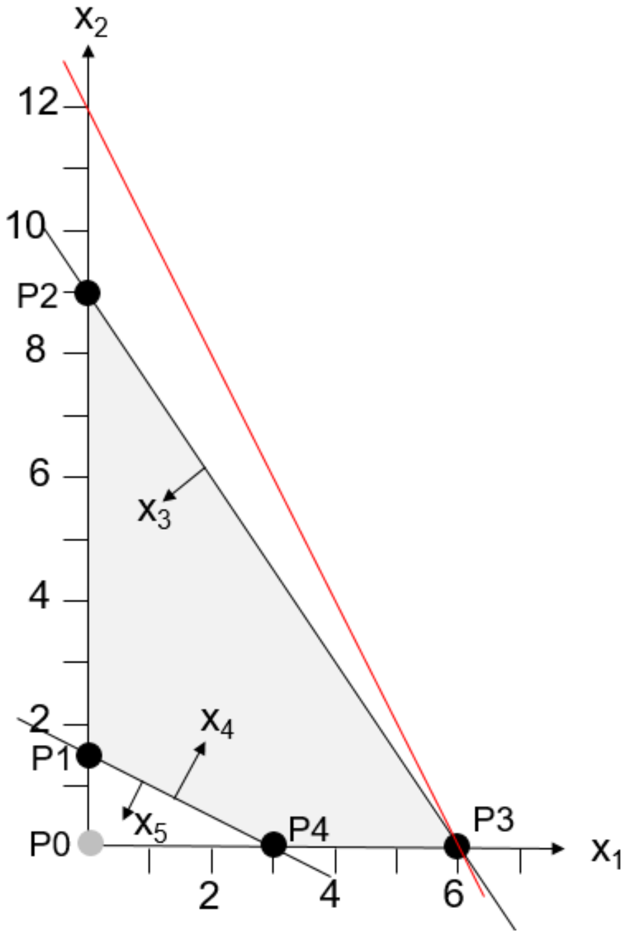
Two-Phase Method: Transformation to Objective Function z



BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	15	2	0	1	1	-1	(7)
x_2	1.5	0.5	$\boxed{1}$	0	-0.5	0.5	(8)
$-w$	0	0	0	0	0	-1	(9)

BV	Value	x_1	x_2	x_3	x_4	x_5

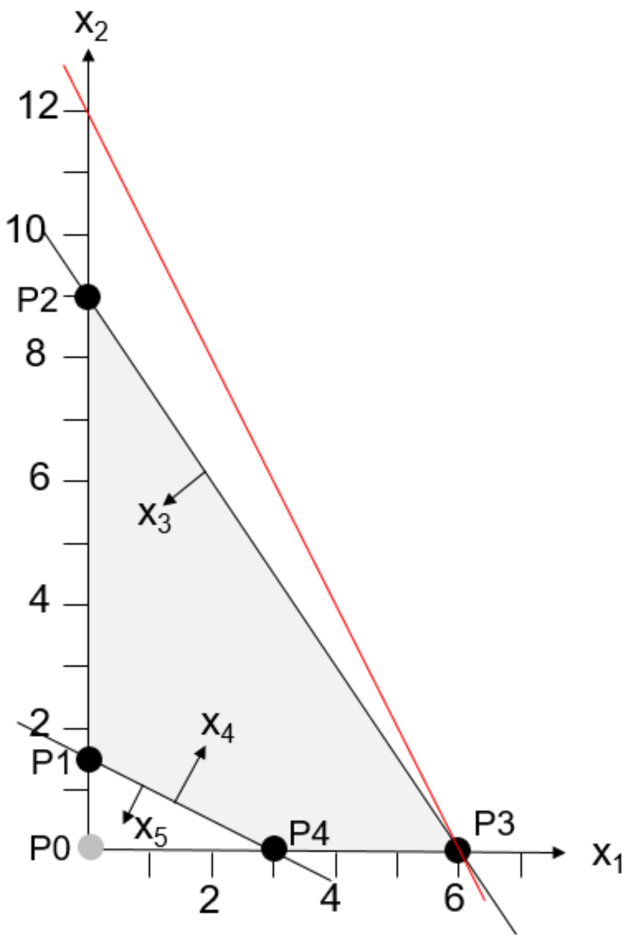
Two-Phase Method: Optimizing Objective Function z



BV	Value	x_1	x_2	x_3	x_4	
x_3	15	2	0	1	1	(7)
x_2	1.5	0.5	1	0	-0.5	(8)
$-z$	0	2	1	0	0	(9)

BV	Value	x_1	x_2	x_3	x_4	x_5

Two-Phase Method: Optimal Objective Function z

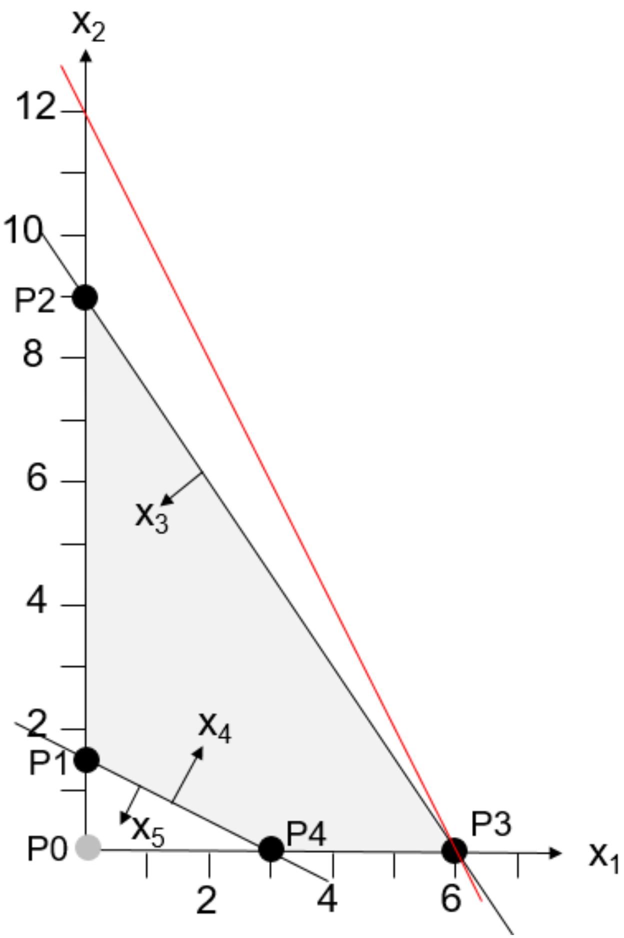


BV	Value	x_1	x_2	x_3	x_4
x_4	3	0	-1.33	0.33	1
x_1	6	1	0.66	0.33	0
$-z$	-12	0	-0.33	-0.66	0

3.4.2

Solving the LP with Artificial Variables: Big M Method

Big M Method: Standard Form and Objective Function

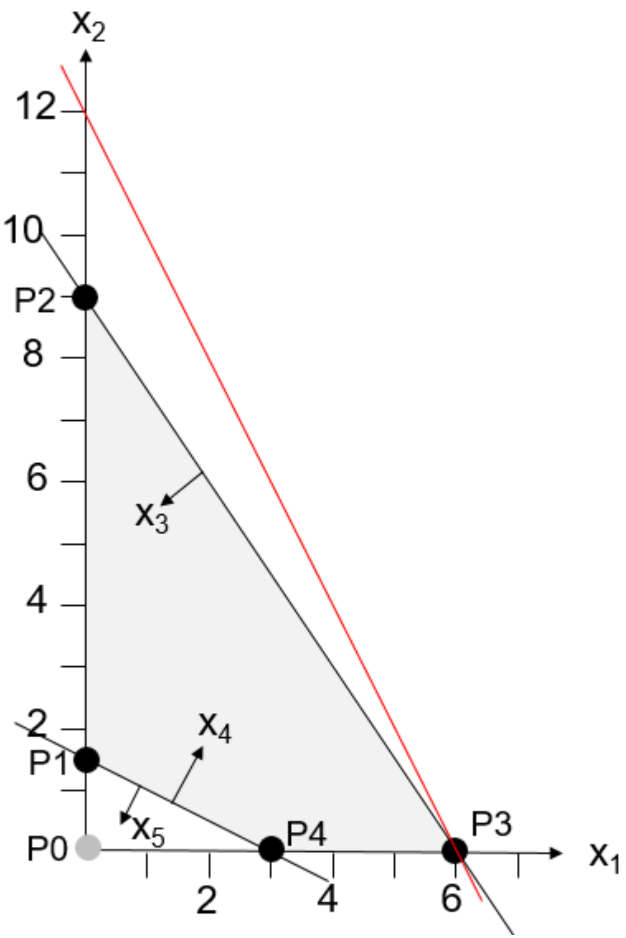


Max $2\ x_1 + \ 1\ x_2$
subject to

$$3\ x_1 + 2\ x_2 \leq 18$$
$$1\ x_1 + 2\ x_2 \geq \ 3$$
$$x_1,\ x_2 \qquad \geq \ 0$$

BV	Value	x_1	x_2	x_3	x_4	x_5

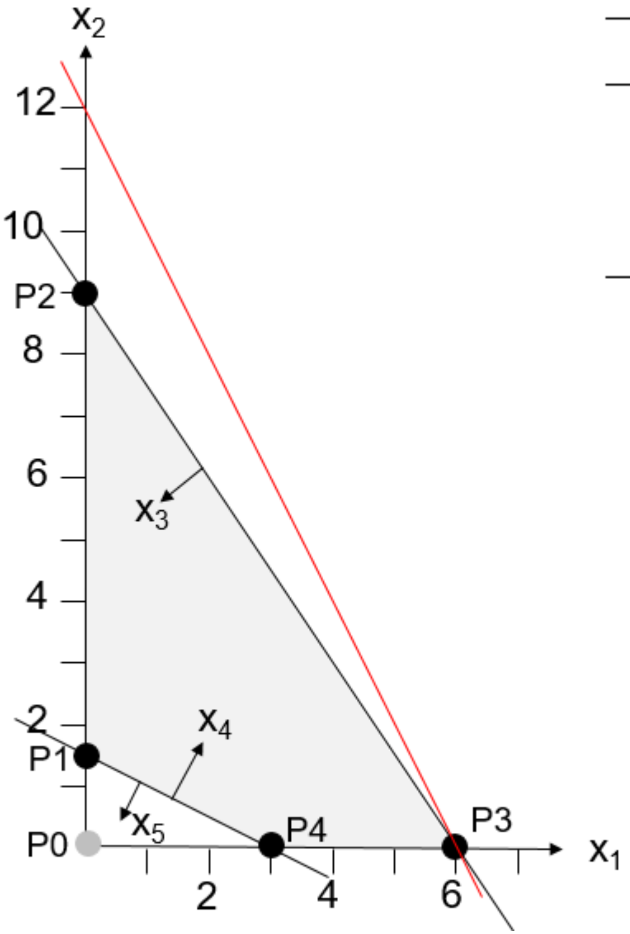
Big M Method: Obtaining Standard Form



BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	18	3	2	1	0	0	(1)
x_5	3	1	2	0	-1	1	(2)
$-z$	0	2	1	0	0	$-M$	(3)

BV	Value	x_1	x_2	x_3	x_4	x_5

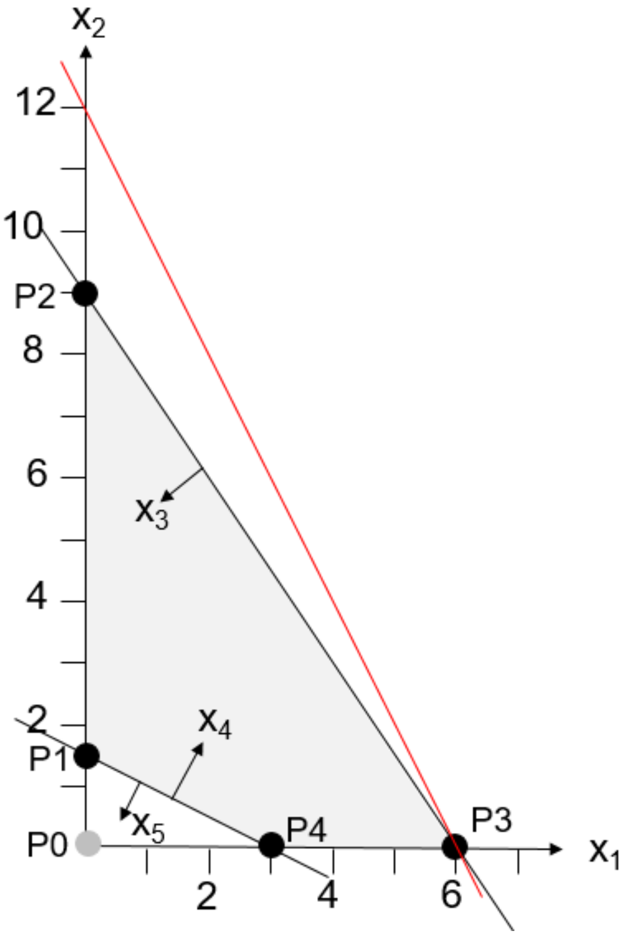
Big M Method: Solving the LP with M in Objective Function



BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	18	3	2	1	0	0	(4)
x_5	3	1	2	0	-1	1	(5)
$-z$	$3 \cdot M$	$2 + M$	$1 + 2 \cdot M$	0	$-M$	0	(6)

BV	Value	x_1	x_2	x_3	x_4	x_5

Big M Method: Elimination of Artificial Variable

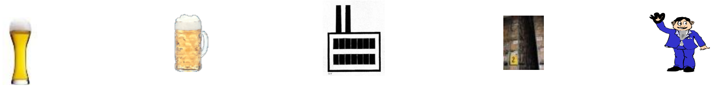


BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	15	2	0	1	1	-1
x_2	1.5	0.5	$\boxed{1}$	0	-0.5	0.5
$-z$	-1.5	1.5	0	0	0.5	$-0.5 - M$

BV	Value	x_1	x_2	x_3	x_4	x_5

3.5 Shadow Prices and Reduced Costs

Shadow Prices
The Beer Glass Problem: Linear Program



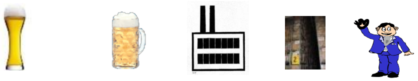
$$\text{Maximize } z = 5x_1 + 7x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$\begin{array}{rcrcrcrcrcl}
6x_1 & + & 5x_2 & + & 1x_3 & + & 0x_4 & + & 0x_5 & = & 60 \\
5x_1 & + & 10x_2 & + & 0x_3 & + & 1x_4 & + & 0x_5 & = & 75 \\
1x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & + & 1x_5 & = & 8 \\
1x_1 & & & & & & & & & \geq & 0 \\
 & & 1x_2 & & & & & & & \geq & 0
\end{array}$$

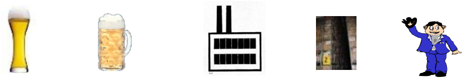
Simplex Tableau Beer Glass Problem

Start tableau:



BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	60	6	5	1	0	0
x_4	75	5	10	0	1	0
x_5	8	1	0	0	0	1
-z	0	5	4.5	0	0	0

Optimal tableau:



BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4.28	0	1	-0.14	0.17	0
x_5	1.57	0	0	-0.28	0.14	1
x_1	6.42	1	0	0.28	-0.14	0
-z	-51.42	0	0	-0.78	-0.05	0

Objective function of the optimal solution $z = 51.42 + 0x_1 + 0x_2 - 0.78x_3 - 0.05x_4 + 0x_5$

Definition Shadow Price

Definition 18: For a linear program the **shadow price** of a constraint is the change in the optimal value of the objective function per unit increase in the right-hand side, all other problem data remaining unchanged.

Alternative names: Dual variable, or opportunity cost.

Positive or Negative Shadow Price?

$$
\begin{array}{llllr}
\text{Max} & z = & & 1\,x_2 & (2.48) \\
\text{subject to} & & & & \\
 & 1\,x_1 + & 1\,x_2 & \leq 15 & (2.49) \\
 & 1\,x_1 & & \geq 10 & (2.50) \\
 & x_1\,, & x_2 & \geq 0 & (2.51)
\end{array}
$$

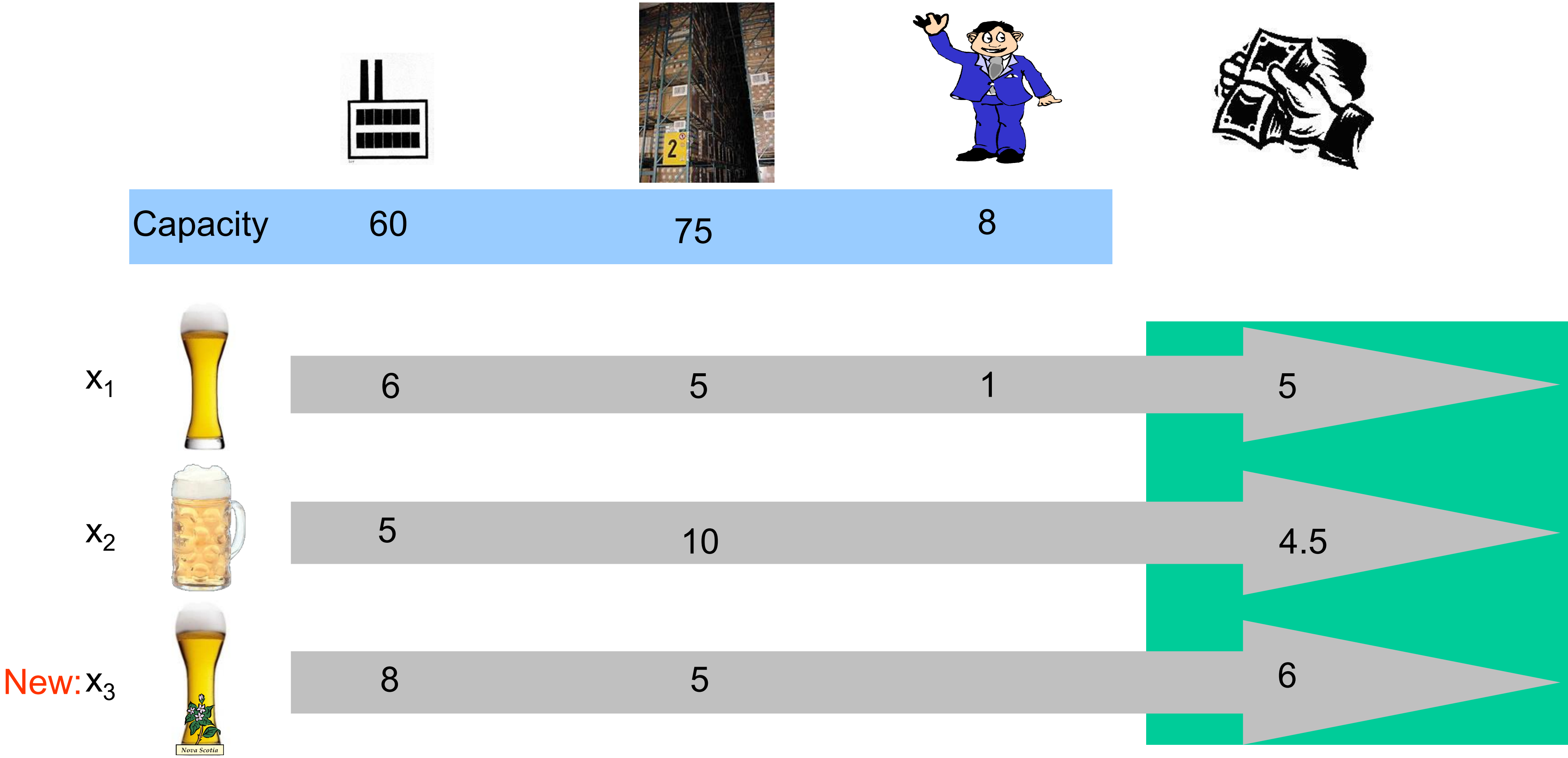
Optimal tableau:

BV	Value	x_1	x_2	x_3	x_4
x_2	5		1	1	1
x_1	10	1			-1
-z	-5			-1	-1

For an increase of the RHS by 1 the objective function changes by the absolute value of the coefficient in the objective function of the associated slack variable. The direction of the change is according to the following table:

Constraint	Objective function	
	Max	Min
≤	+	-
≥	-	+

Reduced Cost
Motivation: The Extended Beer Glass Problem



		Production	Inventory	Demand	Contribution margin
Capacity		60	75	8	
	x_1	6	5	1	5
	x_2	5	10		4.5
New:	x_3	8	5		6

LP Model

Linear Program:

new

$$
\begin{array}{lrrrrrrrl}
\text{Max} & z = & 5\,x_1 & + & 4.5\,x_2 & + & 6\,x_3 & & \\
\text{subject to} & & & & & & & & \\
& & 6\,x_1 & + & 5\,x_2 & + & 8\,x_3 & \leq & 60 \\
& & 5\,x_1 & + & 10\,x_2 & + & 5\,x_3 & \leq & 75 \\
& & 1\,x_1 & & & & & \leq & 8 \\
& & x_1\,, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

Simplex Tableau

First tableau extended beer glass problem:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	60	6	5	8	1	0	0
x_5	75	5	10	5	0	1	0
x_6	8	1	0	0	0	0	1
-z	0	5	4.5	6	0	0	0

Optimal tableau extended beer glass problem:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	4.28	0	1	-0.28	-0.14	0.17	0
x_6	1.57	0	0	-1.57	-0.28	0.14	1
x_1	6.42	1	0	1.57	0.28	-0.14	0
-z	-51.42	0	0	-0.57	-0.78	-0.05	0

Obj. function of the optimal solution $z = 51.42 + 0x_1 + 0x_2 - 0.57x_3 - 0.78x_4 - 0.05x_5 + 0x_6$

Note: The optimal tableau of the extended problem is the optimal tableau of the original problem plus the x_3-column.

Definition Reduced Costs

Definition 19: For a max problem, the reduced cost of a non-basic decision variable is the coefficient in the objective function of the optimal tableau. The reduced cost give the change of the value of the optimal objective function if the value of the variable is increased from 0 to 1.

Note:

- By increasing the value of a non-basic variable by 1 we are forcing it into the basis.
- Each basic variable has reduced cost of 0.

Relationship between Shadow Prices and Reduce Cost: Pricing Out

Definition 20: Calculating the reduced cost of a new decision variable by using the shadow prices from the optimal solution is termed pricing out.

Reduced cost for decision variable j:

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} \cdot y_i$$

with:

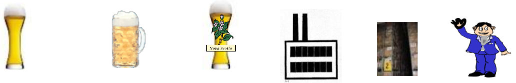
y_i = shadow price of constraint i

c_j = objective function coefficient of decision variable j (often contributed margin)

$a_{i,j}$= coefficient of decision variable j in constraint i (often production coefficient)

Relationship between Shadow Prices and Reduced Cost

First tableau extended beer glass problem:



BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	60	6	5	8	1	0	0
x_5	75	5	10	5	0	1	0
x_6	8	1	0	0	0	0	1
-z	0	5	4.5	6	0	0	0

Optimal tableau extended beer glass problem:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	4.28	0	1	-0.28	-0.14	0.17	0
x_6	1.57	0	0	-1.57	-0.28	0.14	1
x_1	6.42	1	0	1.57	0.28	-0.14	0
-z	-51.42	0	0	-0.57	-0.78	-0.05	0

Reduced Cost: $\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} \cdot y_i$

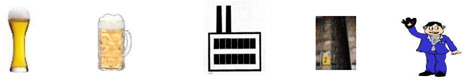
Example Pricing Out

First tableau extended beer glass problem:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	60	6	5	8	1	0	0
x_5	75	5	10	5	0	1	0
x_6	8	1	0	0	0	0	1
-z	0	5	4.5	6	0	0	0

Reduced Cost: $\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} \cdot y_i$

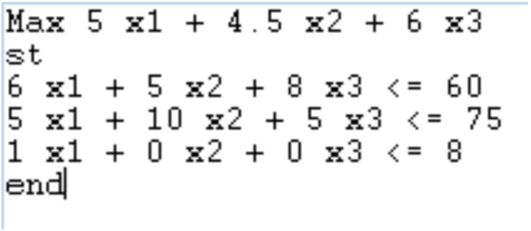
Optimal tableau 2-glass problem:



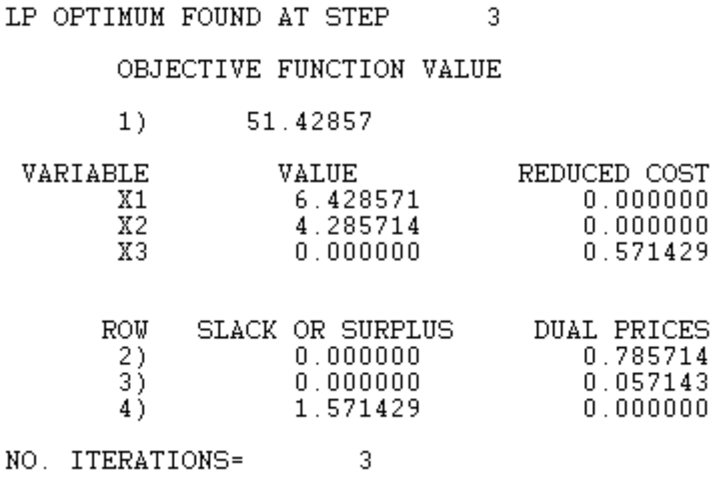
BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4.28	0	1	-0.14	0.17	0
x_6	1.57	0	0	-0.28	0.14	1
x_1	6.42	1	0	0.28	-0.14	0
-z	-51.42	0	0	-0.78	-0.05	0

Calculation of Reduced Costs for the Extended Beer Glass Problem

Model and optimal solution of the extended beer glass problem in LINDO:



```
Max 5 x1 + 4.5 x2 + 6 x3
st
6 x1 + 5 x2 + 8 x3 <= 60
5 x1 + 10 x2 + 5 x3 <= 75
1 x1 + 0 x2 + 0 x3 <= 8
end
```



```
LP OPTIMUM FOUND AT STEP      3

      OBJECTIVE FUNCTION VALUE

      1)      51.42857

 VARIABLE        VALUE          REDUCED COST
       X1         6.428571          0.000000
       X2         4.285714          0.000000
       X3         0.000000          0.571429


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000          0.785714
       3)         0.000000          0.057143
       4)         1.571429          0.000000

NO. ITERATIONS=       3
```

Calculation of reduced costs:

$$\bar{c}_1 = 5 - 0.785713 \cdot 6 + 0.057143 \cdot 5 + 0 \cdot 1 = 0$$

$$\bar{c}_2 = 4.5 - 0.785713 \cdot 5 + 0.057143 \cdot 10 + 0 \cdot 0 = 0$$

$$\bar{c}_3 = 6 - 0.785713 \cdot 8 + 0.057143 \cdot 5 + 0 \cdot 0 = -0.57$$

Note: that the shadow prices of the extended beer glass problem are the same as for the beer glass problem with two products and hence we can calculate the reduced cost for variable x3 although it is not part beer glass problem with two products.

3.6 Duality in Linear Programming

3.6.1 Motivation

Duality: Motivation

Car production planning problem

Linear Program

	Trend	Comfort	Sport		
Max $z =$	$6 x_1$	$+ 14 x_2$	$+ 13 x_3$		
subject to					
	$0.5 x_1$	$+ 2 x_2$	$+1 x_3 \leq 24$		Metal shop
	$1 x_1$	$+ 2 x_2$	$+ 4 x_3 \leq 60$		Wood shop
			$x_j \geq 0$		

Optimal tableau

BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	-11	-0.5

Car Production Problem: The Value of Capacity

Start tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	24	½	2	1	1	0
x_5	60	1	2	4	0	1
-z	0	6	14	13	0	0

Optimal tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	-11	-0.5

Objective function value:

- Primal perspective: Contributions of products (revenues)
- Dual perspective: Value of capacity (shadow prices)

From the Primal LP …

Primal LP:

$$\text{Max } z = 6\,x_1 + 14\,x_2 + 13\,x_3 \qquad (1.57)$$

subject to

$$0.5\,x_1 + 2\,x_2 + 1\,x_3 \leq 24 \qquad (1.58)$$

$$1\,x_1 + 2\,x_2 + 4\,x_3 \leq 60 \qquad (1.59)$$

$$x_j \geq 0 \qquad (1.60)$$

… to the Dual LP

$$
\begin{aligned}
\text{Min } v = \; & 24 \cdot y_1 + 60 \cdot y_2 \\
\text{subject to} \; & \\
& 0.5 \cdot y_1 + 1 \cdot y_2 \geq 6 \\
& 2 \cdot y_1 + 2 \cdot y_2 \geq 14 \\
& 1 \cdot y_1 + 4 \cdot y_2 \geq 13 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

Set prices such that the value of capacity is minimized

Cost of product ≥ contribution of product (Optimality condition of Simplex)

Prices are non-negative

Relation Between Primal and Dual LP

Property 3**:** For every primal program there exist a dual program. The dual program of a dual program is the (original) primal program.

Optimal Tableaus of the Primal and the Dual LP

Primal LP DV

$\text{Max } z = 6x_1 + 14x_2 + 13x_3$

subject to

$0.5x_1 + 2x_2 + 1x_3 \leq 24$ y_1

$1x_1 + 2x_2 + 4x_3 \leq 60$ y_2

$x_j \geq 0$

Dual LP DV

$\text{Min } v = 24y_1 + 60y_2$

subject to

$0.5y_1 + 1y_2 \geq 6$ x_1

$2y_1 + 2y_2 \geq 14$ x_2

$1y_1 + 4y_2 \geq 13$ x_3

$y_1, y_2 \geq 0$

Optimal tableau of the primal LP:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	-11	-0.5
x*		36	0	6	0	0

Optimal tableau of the dual LP:

BV	Value	y_1	y_2	y_3	y_4	y_5
y_4	9	0	0	-6	1	1
y_1	11	1	0	-4	0	1
y_2	0.5	0	1	1	0	-0.5
-v	-294	0	0	-36	0	-6
y*		11	0	0	9	0

3.6.2 Rules for Obtaining the Dual LP

Correspondence between Primal and Dual Problem

Maximizing ↔	Minimizing
i-th constraint ≤	i-th variable ≥ 0
i-th constraint ≥	i-th variable ≤ 0
i-th constraint =	i-th variable is real
j-th variable ≥ 0	j-th constraint ≥
j-th variable ≤ 0	j-th constraint ≤
j −th variable real	j-th constraint =

Obtaining the Dual: Example

Primal LP

$$\text{Max } z = -1 \cdot x_1 + 1 \cdot x_2 - 2 \cdot x_3$$

subject to

$$1 \cdot x_1 + 1 \cdot x_2 + 3 \cdot x_3 = 7$$

$$2 \cdot x_1 - 3 \cdot x_2 + 1 \cdot x_3 \leq 15$$

$$1 \cdot x_1 + 2 \cdot x_2 - 1 \cdot x_3 \geq -3$$

$$2 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \leq 12$$

$$x_1 \in R$$

$$x_2 \leq 0$$

$$x_3 \geq 0$$

Dual LP

Maximizing ↔	Minimizing
i-th constraint ≤	i-th variable ≥ 0
i-th constraint ≥	i-th variable ≤ 0
i-th constraint =	i-th variable is real
j-th variable ≥ 0	j-th constraint ≥
j-th variable ≤ 0	j-th constraint ≤
j-th variable real	j-th constraint =

Recap: Algebraic Sign of Shadow Prices

The algebraic sign of the shadow price equals the domain of a dual variable.

Example: For a “≤”-constraint and a “max” objective function we obtain a “≥” dual variable (“+” sign in the Table on the left).

Table given on Slide 77:

Constraint	Objective	
	Max	Min
≤	+	-
≥	-	+

Max objective function ↔	Min objective function
i-th constraint ≤	i-th variable ≥ 0
i-th constraint ≥	i-th variable ≤ 0
j-th variable ≥ 0	j-th constraint ≥
j-th variable ≤ 0	j-th constraint ≤

Weak and Strong Duality Property

Property 4: Weak Duality Property

If x with objective function value z(x) is a feasible but not optimal solution of the primal LP and y with objective function value v(y) is a feasible but not optimal solution of the dual LP, then

$$v(y) > z(x)$$

holds.

Property 5: Strong Duality Property

If x* with objective function value z(x*) is a feasible and optimal solution of the primal LP and y* with objective function value v(y*) is a feasible and optimal solution of the dual LP, then

$$v(y^*) = z(x^*)$$

holds.

Weak and Strong Duality Property: Example

Primal LP

Max z = $6 x_1 + 14 x_2 + 13 x_3$

s.t.

$$0.5 x_1 + 2 x_2 + 1 x_3 \leq 24$$

$$1 x_1 + 2 x_2 + 4 x_3 \leq 60$$

$$x_j \geq 0$$

Dual LP

Min v = $24 \cdot y_1 + 60 \cdot y_2$

s.t.

$$0.5 \cdot y_1 + 1 \cdot y_2 \geq 6$$

$$2 \cdot y_1 + 2 \cdot y_2 \geq 14$$

$$1 \cdot y_1 + 4 \cdot y_2 \geq 13$$

$$y_1, y_2 \geq 0$$

Weak duality property:

Feasible solution for the primal LP:

Feasible solution for the dual LP:

Strong duality property:

Optimal primal tableau

BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	**-11**	**-0.5**
x*		36	0	6	0	0

Optimal dual tableau

BV	Value	y_1	y_2	y_3	y_4	y_5
y_4	9	0	0	-6	1	1
y_1	11	1	0	-4	0	1
y_2	0.5	0	1	1	0	-0.5
-v	-294	0	0	**-36**	**0**	**-6**
y*		11	0.5	0	9	0

Unboundness Property

Property 6: Unboundedness Property

If the primal problem has an unbounded solution, then the dual problem is infeasible.

Complementary Slackness Property

Property 7: Complementary Slackness Property

With x* be a feasible and optimal solution of the primal problem and y* be a feasible and optimal solution of the dual problem, the following holds:

$$(1) \quad \text{If} \quad x^*_j > 0, \quad \text{then} \quad \sum_{i=1}^{m} a_{ij} \cdot y^*_i = c_j \quad \text{and thus } y^*_{m+j} = 0$$

That is, if the variable of the primal problem is in the basis, the slack variable of the associated constraint in the dual problem is zero.

Complementary Slackness Property: Example

Primal LP

$\text{Max } z = 6\,x_1 \quad + 14\,x_2 \quad + 13\,x_3$

s.t.

$$0.5\,x_1 + 2\,x_2 + 1\,x_3 \leq 24$$

$$1\,x_1 + 2\,x_2 + 4\,x_3 \leq 60$$

$$x_j \geq 0$$

Dual LP

$\text{Min } v = 24 \cdot y_1 \quad + 60 \cdot y_2$ (Surplus Var.)

s.t.

$$0.5 \cdot y_1 + 1 \cdot y_2 \geq 6 \qquad (y_3)$$

$$2 \cdot y_1 + 2 \cdot y_2 \geq 14 \qquad (y_4)$$

$$1 \cdot y_1 + 4 \cdot y_2 \geq 13 \qquad (y_5)$$

$$y_1,\ y_2 \geq 0$$

Optimal primal tableau

BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	**-11**	**-0.5**
x*		36	0	6	0	0

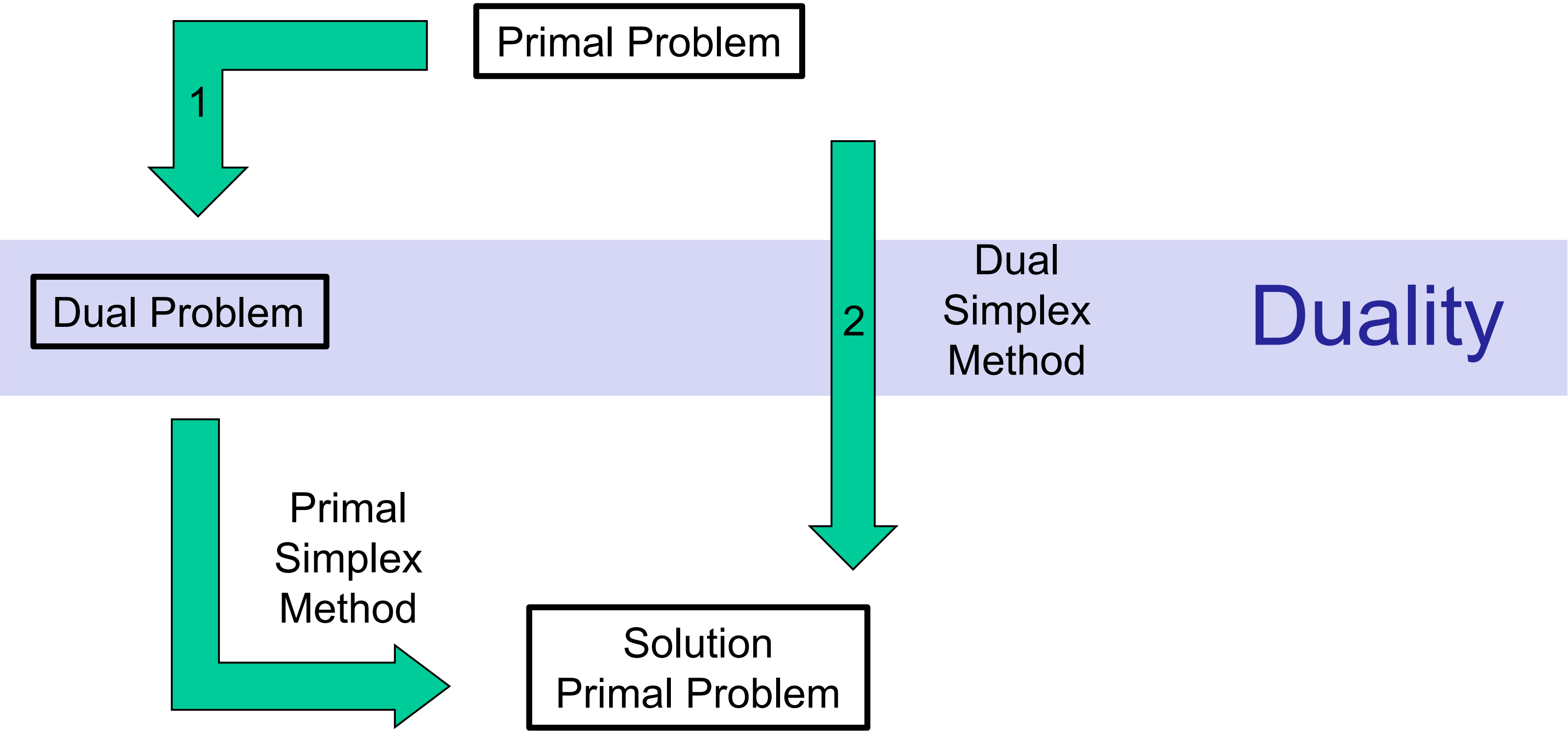
Optimal dual tableau

BV	Value	y_1	y_2	y_3	y_4	y_5
y_4	9	0	0	-6	1	1
y_1	11	1	0	-4	0	1
y_2	0.5	0	1	1	0	-0.5
-v	-294	0	0	**-36**	**0**	**-6**
y*		11	0.5	0	9	0

3.7 The Dual Simplex Method

3.7.1 Motivation of the Dual Simplex Method

Motivation of the Dual Simplex Method



Primal Problem

1

Dual Problem

2

Dual Simplex Method

Duality

Primal Simplex Method

Solution Primal Problem

Example for Approach 1: Obtaining the Dual LP

Primal LP:

Dual variables

$$\begin{array}{llllll}
\text{Max } z = & -1\cdot x_1 & -1\cdot x_2 & & & \\
& \text{s.t.} & & & & \\
& -2\cdot x_1 & -1\cdot x_2 & \leq & 4 & (y_1 \geq 0) \\
& -2\cdot x_1 & +4\cdot x_2 & \leq & -8 & (y_2 \geq 0) \\
& -1\cdot x_1 & +3\cdot x_2 & \leq & -7 & (y_3 \geq 0) \\
& & x_1, x_2 & \geq & 0 &
\end{array}$$

(1) (2) Dual constraints

Dual LP:

$$\begin{array}{lllllll}
& \text{Min } v = & 4\cdot y_1 & -8\cdot y_2 & -7\cdot y_3 & & \\
& & \text{s.t.} & & & & \\
(1) & & -2\cdot y_1 & -2\cdot y_2 & -1\cdot y_3 & \geq & -1 \\
(2) & & -1\cdot y_1 & +4\cdot y_2 & +3\cdot y_3 & \geq & -1 \\
& & & & y_1, y_2, y_3 & \geq & 0
\end{array}$$

Transforming the Dual LP into Standard Form

$$
\begin{array}{lrrrcr}
\text{Min } v= & 4\cdot y_1 & -8\cdot y_2 & -7\cdot y_3 & & \\
 & \text{s.t.} & & & & \\
 & -2\cdot y_1 & -2\cdot y_2 & -1\cdot y_3 & \geq & -1 \\
 & -1\cdot y_1 & +4\cdot y_2 & +3\cdot y_3 & \geq & -1 \\
 & & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

Correspondence between Primal and Dual: Start-Tableau

Dual Problem:

$$
\begin{array}{lllll}
\text{Max } -v = & -4\cdot y_1 & +8\cdot y_2 & +7\cdot y_3 & & \\
\text{s.t.} & & & & & \\
& +2\cdot y_1 & +2\cdot y_2 & +1\cdot y_3 & \leq & 1 \\
& +1\cdot y_1 & -4\cdot y_2 & -3\cdot y_3 & \leq & 1 \\
& & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

BV	**Value**	y_1	y_2	y_3	y_4	y_5
y_4	1	2	2	1	1	
y_5	1	1	- 4	- 3		1
v	0	- 4	8	7	0	0

Primal Problem:

$$
\begin{array}{llllll}
\text{Max } z = & -1\cdot x_1 & -1\cdot x_2 & & \\
\text{s.t.} & & & & \\
& -2\cdot x_1 & -1\cdot x_2 & \leq & 4 \\
& -2\cdot x_1 & +4\cdot x_2 & \leq & -8 \\
& -1\cdot x_1 & +3\cdot x_2 & \leq & -7 \\
& & x_1, x_2 & \geq & 0
\end{array}
$$

BV	**Value**	x_1	x_2	x_3	x_4	x_5
x_3	4	- 2	- 1	1		
x_4	- 8	- 2	4		1	
x_5	- 7	- 1	3			1
- z	0	- 1	- 1	0	0	0

Primal Optimality and Dual Feasibility

Dual Problem:

BV	Value	y_1	y_2	y_3	y_4	y_5
y_4	1	2	2	1	1	
y_5	1	1	- 4	- 3		1
v	0	- 4	8	7	0	0

Primal Problem:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	4	- 2	- 1	1		
x_4	- 8	- 2	4		1	
x_5	- 7	- 1	3			1
- z	0	- 1	- 1	0	0	0

Primal Infeasibility and Dual Non-Optimality

Dual Problem:

BV	Value	y_1	y_2	y_3	y_4	y_5
y_4	1	2	2	1	1	
y_5	1	1	- 4	- 3		1
v	0	- 4	8	7	0	0

Primal Problem:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	4	- 2	- 1	1		
x_4	- 8	- 2	4		1	
x_5	- 7	- 1	3			1
- z	0	- 1	- 1	0	0	0

Applying the Primal Simplex to the Dual Problem

Dual problem:

BV	Value	y_1	y_2	y_3	y_4	y_5
y_4	1	2	2	1	1	0
y_5	1	1	- 4	- 3	0	1
v	0	- 4	8	7	0	0

Primal problem:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	4	- 2	- 1	1	0	0
x_4	- 8	- 2	4	0	1	0
x_5	- 7	- 1	3	0	0	1
- z	0	- 1	- 1	0	0	0

Tableaus After the First Pivot Step

Dual problem:

BV	Value	y_1	y_2	y_3	y_4	y_5
y_2	0.5	1	$\boxed{1}$	0.5	0.5	0
y_5	3	5	0	- 1	2	1
v	- 4	-12	0	3	-4	0

Primal problem:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	12	0	- 5	1	- 1	0
x_1	4	$\boxed{1}$	- 2	0	- 0.5	0
x_5	- 3	0	1	0	- 0.5	1
- z	4	0	-3	0	- 0.5	0

Next Pivot Step

Dual problem:

BV	**Value**	y_1	y_2	y_3	y_4	y_5
y_2	0.5	1	1	0.5	0.5	
y_5	3	5		- 1	2	1
v	- 4	- 12		3	-4	

Primal problem:

BV	**Value**	x_1	x_2	x_3	x_4	x_5
x_3	12		- 5	1	- 1	
x_1	4	1	- 2		- 0.5	
x_5	- 3		1		- 0.5	1
- z	4		-3		- 0.5	

Optimal Tableaus

Dual problem:

BV	Value	y_1	y_2	y_3	y_4	y_5
y_3	1	2	2	1	1	0
y_5	4	7	2	0	3	1
v	- 7	- 18	- 6	0	- 7	0

Primal problem:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	18	0	-7	1	0	-2
x_1	7	1	-3	0	0	-1
x_4	6	0	-2	0	1	- 2
-z	7	0	-4	0	0	- 1

Dual Simplex: Choosing the Pivot Element

Example:

BV	Value	x_1	x_2	x_3	x_4	
x_3	-1	-1	-1	1	0	(1)
x_4	-2	-2	-3	0	1	(2)
-z	0	-3	-1	0	0	(3)

3.7.2 Formal Presentation of the Dual Simplex

Formal Presentation of the Dual Simplex Method

Condition:
LP is dual feasible ($\bar{c}_j \leq 0$ for all j) and dual non-optimal ($\bar{b}_i < 0$ for at least one i).

1. If $\bar{b}_i \geq 0, \ \forall \ i = 1, \ldots, m$ and $\bar{c}_j \leq 0, \ \forall \ j = 1, \ldots, n,$ then stop:

   Optimal solution.

   If there exists some $\bar{b}_i < 0,$ then continue.

m = number of constraints

i = constraint index

$\bar{b}_i$= current right-hand-side

n = number of variables

j = variable index

$\bar{c}_j$ = current coefficient objective function value

Formal Presentation of the Dual Simplex Method

1. If $\bar{b}_i \geq 0, \; \forall \; i = 1, \ldots, m$ and $\bar{c}_j \leq 0, \; \forall \; j = 1, \ldots, n,$ then stop:

   Optimal solution.

   If there exists some $\bar{b}_i < 0,$ then continue.

2. Choose pivot row r, such that

$$\bar{b}_r = \min_{i=1}^{m} \left\{ \bar{b}_i : \bar{b}_i < 0 \right\}$$

   If $\bar{a}_{rj} \geq 0, \forall j = 1, \ldots, n,$ then stop:

   No feasible solution.

   If there exists $\bar{a}_{rj} < 0,$ for some $j = 1, \ldots, n,$ then continue.

m = number of constraints

i = constraint index

$\bar{b}_i$= current right-hand-side

n = number of variables

j = variable index

$\bar{c}_j$ = current coefficient objective function value

$\bar{a}_{r,j}$= current element (r,j) in matrix

Formal Presentation of the Dual Simplex Method

1. If $\bar{b}_i \geq 0, \; \forall \; i = 1, \ldots, m$ and $\bar{c}_j \leq 0, \; \forall \; j = 1, \ldots, n,$ then stop:

   Optimal solution.

   If there exists some $\bar{b}_i < 0,$ then continue.

2. Choose pivot row r, such that

$$\bar{b}_r = \min_{i=1}^{m} \left\{ \bar{b}_i : \bar{b}_i < 0 \right\}$$

   If $\bar{a}_{rj} \geq 0, \forall j = 1, \ldots, n,$ then stop:

   No feasible solution.

   If there exists $\bar{a}_{rj} < 0,$ for some $j = 1, \ldots, n,$ then continue.

3. Choose pivot column s, such that

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \min_{j=1}^{n} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : \bar{a}_{rj} < 0 \right\}$$

m = number of constraints

i = constraint index

$\bar{b}_i$= current right-hand-side

n = number of variables

j = variable index

c_j = current coefficient objective function value

$\bar{a}_{r,j}$= current element (r,j) in matrix

Formal Presentation of the Dual Simplex Method

1. If $\bar{b}_i \geq 0, \ \forall \ i = 1, \ldots, m$ and $\bar{c}_j \leq 0, \ \forall \ j = 1, \ldots, n,$ then stop:

   Optimal solution.

   If there exists some $\bar{b}_i < 0,$ then continue.

2. Choose pivot row r, such that

$$\bar{b}_r = \min_{i=1}^{m} \left\{ \bar{b}_i : \bar{b}_i < 0 \right\}$$

   If $\bar{a}_{rj} \geq 0, \forall j = 1, ..., n,$ then stop:

   No feasible solution.

   If there exists $\bar{a}_{rj} < 0,$ for some $j = 1, ..., n,$ then continue.

3. Choose pivot column $s,$ such that

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \min_{j=1}^{n} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : \bar{a}_{rj} < 0 \right\}$$

4. Replace the basic variable and re-establish canonical form.

m = number of constraints

i = constraint index

$\bar{b}_i$= current right-hand-side

n = number of variables

j = variable index

$\bar{c}_j$ = current coefficient objective function value

$\bar{a}_{r,j}$= current element (r,j) in matrix

Formal Presentation of the Dual Simplex Method

1. If $\bar{b}_i \geq 0, \ \forall \ i = 1, \ldots, m$ and $\bar{c}_j \leq 0, \ \forall \ j = 1, \ldots, n,$ then stop:

   Optimal solution.

   If there exists some $\bar{b}_i < 0,$ then continue.

2. Choose pivot row r, such that

$$\bar{b}_r = \min_{i=1}^{m} \left\{ \bar{b}_i : \bar{b}_i < 0 \right\}$$

   If $\bar{a}_{rj} \geq 0, \forall j = 1, \ldots, n,$ then stop:

   No feasible solution.

   If there exists $\bar{a}_{rj} < 0,$ for some $j = 1, \ldots, n,$ then continue.

3. Choose pivot column $s,$ such that

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \min_{j=1}^{n} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : \bar{a}_{rj} < 0 \right\}$$

4. Replace the basic variable and re-establish canonical form.

5. Go to 1.

m = number of constraints

i = constraint index

$\bar{b}_i$= current right-hand-side

n = number of variables

j = variable index

$\bar{c}_j$ = current coefficient objective function value

$\bar{a}_{r,j}$= current element (r,j) in matrix

Formal Presentation of the Dual Simplex Method

1. If $\bar{b}_i \geq \ 0, \ \forall \ i = 1, \ldots, m$ and $\bar{c}_j \leq 0, \ \forall \ j = 1, \ldots, n,$ then stop:

   Optimal solution.

   If there exists some $\bar{b}_i < \ 0,$ then continue.

2. Choose pivot row r, such that

$$\bar{b}_r = \min_{i=1}^{m} \left\{ \bar{b}_i : \bar{b}_i < 0 \right\}$$

   If $\bar{a}_{rj} \geq 0, \forall j = 1, \ldots, n,$ then stop:

   No feasible solution.

   If there exists $\bar{a}_{rj} < 0,$ for some $j = 1, \ldots, n,$ then continue.

3. Choose pivot column $s,$ such that

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \min_{j=1}^{n} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : \bar{a}_{rj} < 0 \right\}$$

4. Replace the basic variable and re-establish canonical form.

5. Go to 1.

Property 6: The dual Simplex finishes with

1. an optimal solution
   or
2. a proof, that there is no feasible solution

Applying the Dual Simplex after Adding a new Constraint

Car production planning problem:

Trend Comfort Sport

$$\text{Max } z = 6x_1 + 14x_2 + 13x_3$$

subject to

$$0.5x_1 + 2x_2 + 1x_3 \leq 24 \quad \text{Metal shop}$$

$$1x_1 + 2x_2 + 4x_3 \leq 60 \quad \text{Wood shop}$$

Additional constraint:
Minimum sales comfort (MSC): $x_2 \geq 5$

Optimal primal LP:

		T	C	S	M	W
BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	-11	-0.5

		T	C	S	M	W
BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6	0	4	-1
x_3	6	0	-1	1	-1	0.5
-z	-294	0	-9	0	-11	-0.5

Example Dual Simplex Method

		T	C	S	M	W	MSC
BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_1	36	1	6	0	4	-1	
x_3	6	0	-1	1	-1	1/2	
x_6	-5		-1				1
-z	-294	0	-9	0	-11	-1/2	

		T	C	S	M	W	MSC
BV	Value	x_1	x_2	x_3	x_4	x_5	x_6

3.8 Special Cases

3.8.1 No Feasible Solution

Beer glass example with additional constraint

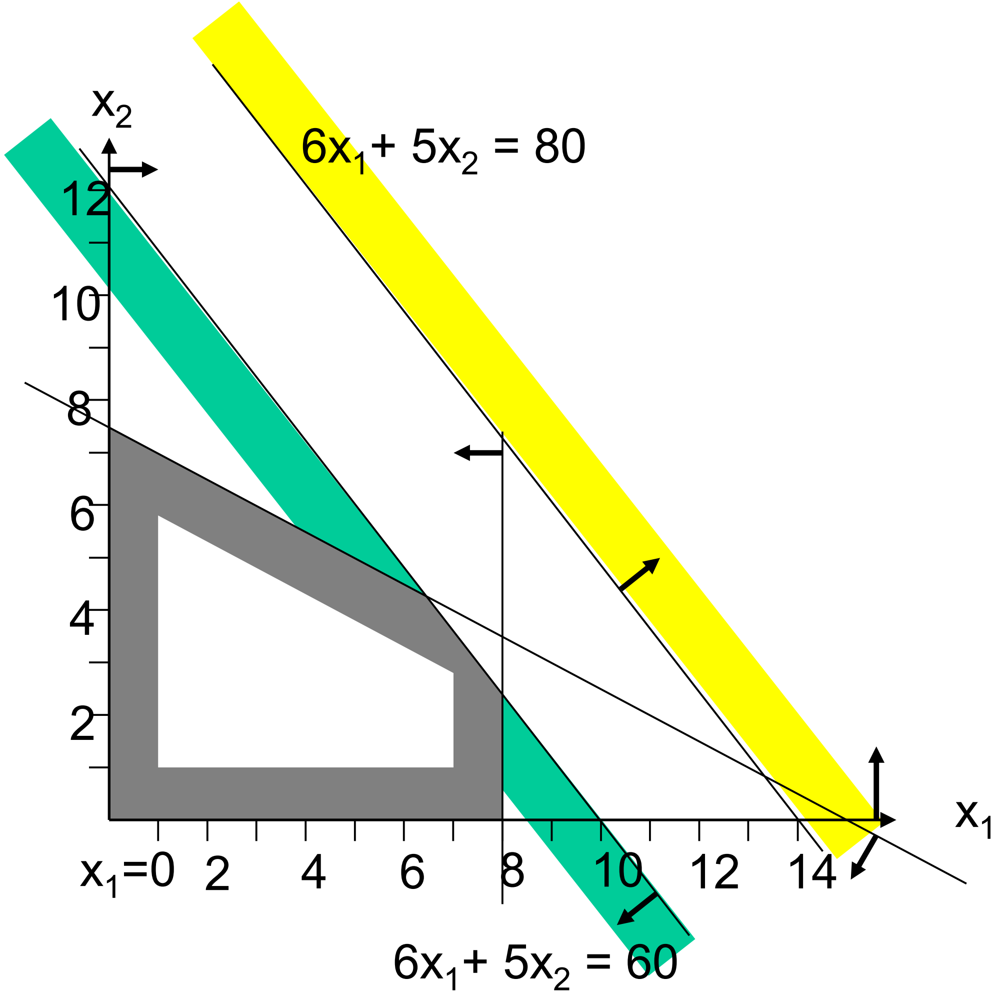
$$
\begin{aligned}
\text{Max } z = 5\,x_1 + 4.5\,x_2 \\
\text{s.t.} \\
6\,x_1 + 5\,x_2 &\leq 60 \\
6\,x_1 + 5\,x_2 &\geq 80 \\
5\,x_1 + 10\,x_2 &\leq 75 \\
1\,x_1 &\leq 8 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

No Feasible Solution

Max z = $5x_1 + 4.5x_2$

s.t.

$$\begin{array}{rcrcr} 6x_1 & + & 5x_2 & \le & 60 \\ 6x_1 & + & 5x_2 & \ge & 80 \\ 5x_1 & + & 10x_2 & \le & 75 \\ 1x_1 & & & \le & 8 \\ x_1, & & x_2 & \ge & 0 \end{array}$$



How to Spot an Infeasible LP in the Tableau

$$
\begin{array}{llr}
\text{Max } z = & 5x_1 + 4.5x_2 - Mx_5 & \\
\text{s.t.} & 6x_1 + 5x_2 + x_3 & = 60 \\
 & 6x_1 + 5x_2 - x_4 + x_5 & = 80 \\
 & 5x_1 + 10x_2 + x_6 & = 75 \\
 & 1x_1 + x_7 & = 8 \\
 & x_1, \ldots, x_7 \geq 0 &
\end{array}
$$

If any artificial variable is positive in the optimal Big M tableau, the original LP has no feasible solution.

Optimal Tableau (Big M Method)

BV	RHS	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_2	4.29	0	1	-0.14	0	0	0.09	0
x_5	20	0	0	-1	-1	1	0	0
x_7	1.58	0	0	-0.29	0	0	0.07	1
x_1	6.42	1	0	0.29	0	0	-0.07	0
-z	20M-62.14	0	0	-0.79-M	-M	0	-0.03	0

3.8.2 Unbounded LP

Beer glass example with ≥-constraints instead of ≤-constraints

$$\text{Max } z = 5\,x_1 + 4.5\,x_2$$

s.t.

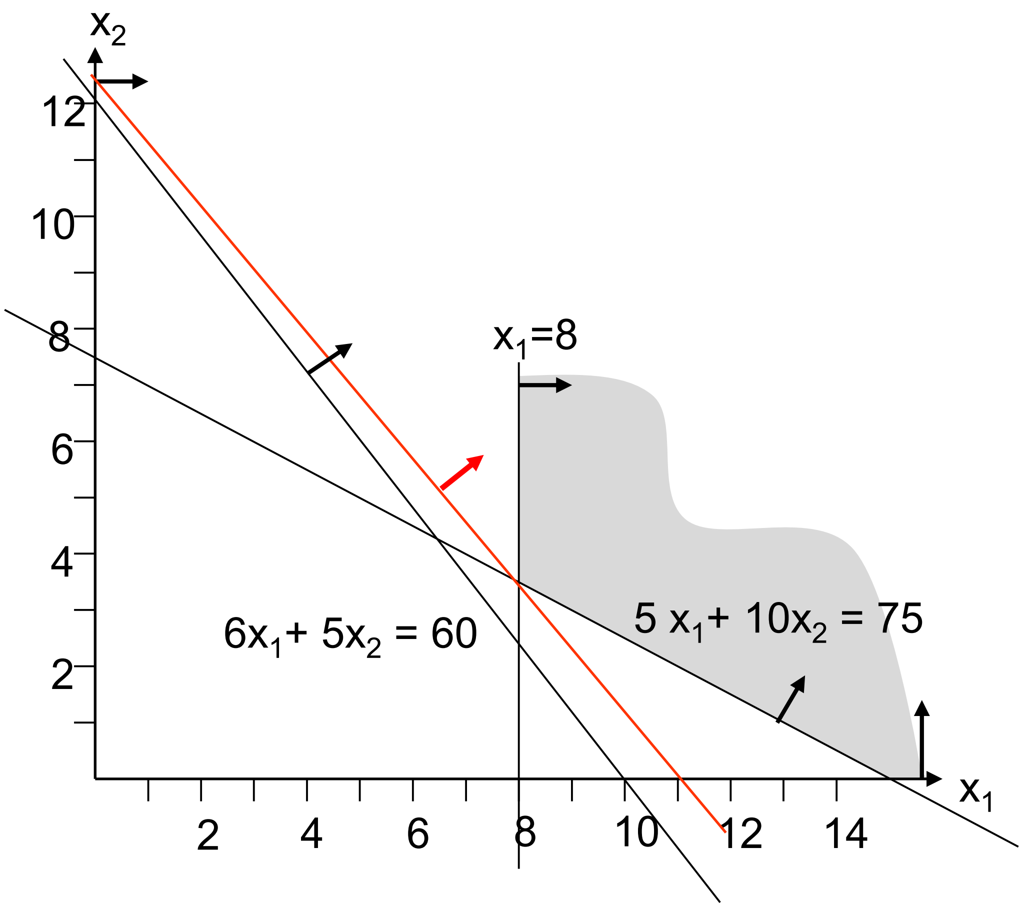
$$\begin{aligned} 6\,x_1 + 5\,x_2 &\geq 60 \\ 5\,x_1 + 10\,x_2 &\geq 75 \\ 1\,x_1 &\geq 8 \\ x_1,\quad x_2 &\geq 0 \end{aligned}$$

Unbounded LP

$$\text{Max } z = 5\,x_1 + 4.5\,x_2$$

s.t.

$$\begin{aligned} 6\,x_1 + 5\,x_2 &\geq 60 \\ 5\,x_1 + 10\,x_2 &\geq 75 \\ 1\,x_1 &\geq 8 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$



Unbounded LP: Example

$$\text{Max } z = 1x_1 + 1x_2 \quad (1)$$

s.t.

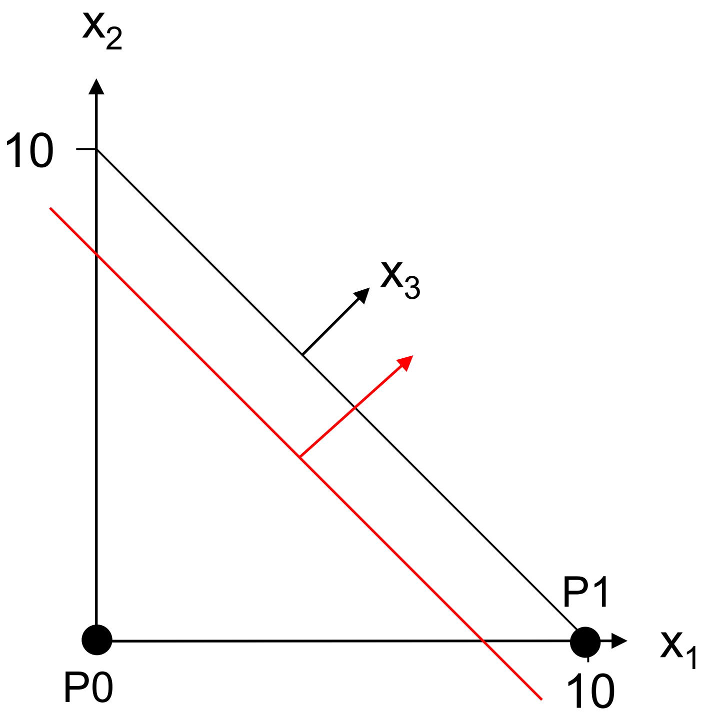
$$1x_1 + 1x_2 \geq 10 \quad (2)$$

$$x_1, \; x_2 \geq 0 \quad (3)$$

$$\text{Max } z = 1x_1 + 1x_2 - Mx_4 \quad (1)$$

s.t.

$$1x_1 + 1x_2 - 1x_3 + 1x_4 = 10 \quad (2)$$



BV	Value	x_1	x_2	x_3
x_1	10	1	1	-1
-z	-10	0	0	1

How to Spot an Unbounded LP

Apply the primal simplex method:

BV	Value	x_1	x_2	x_3	x_4	x_5	
x_3	30	0	7	1	-0.6	0	(7)
x_1	15	1	2	0	-0.1	0	(5)
x_5	7	0	2	0	-0.1	1	(6)
-z	-7500	0	-550	0	50	0	(8)

(2) **Choose pivot column s**.

$$\bar{c}_s = \max_{j=1}^{n} \{ \bar{c}_j \mid \bar{c}_j > 0 \}$$

If $\bar{a}_{i,s} \leq 0$ für i=1,…,m, stop;
Special case: Unbounded solution.
Continue only if there is at least one $\bar{a}_{i,s} > 0$.

3.8.3 Redundant Constraint

Beer glass problem with additional constraint

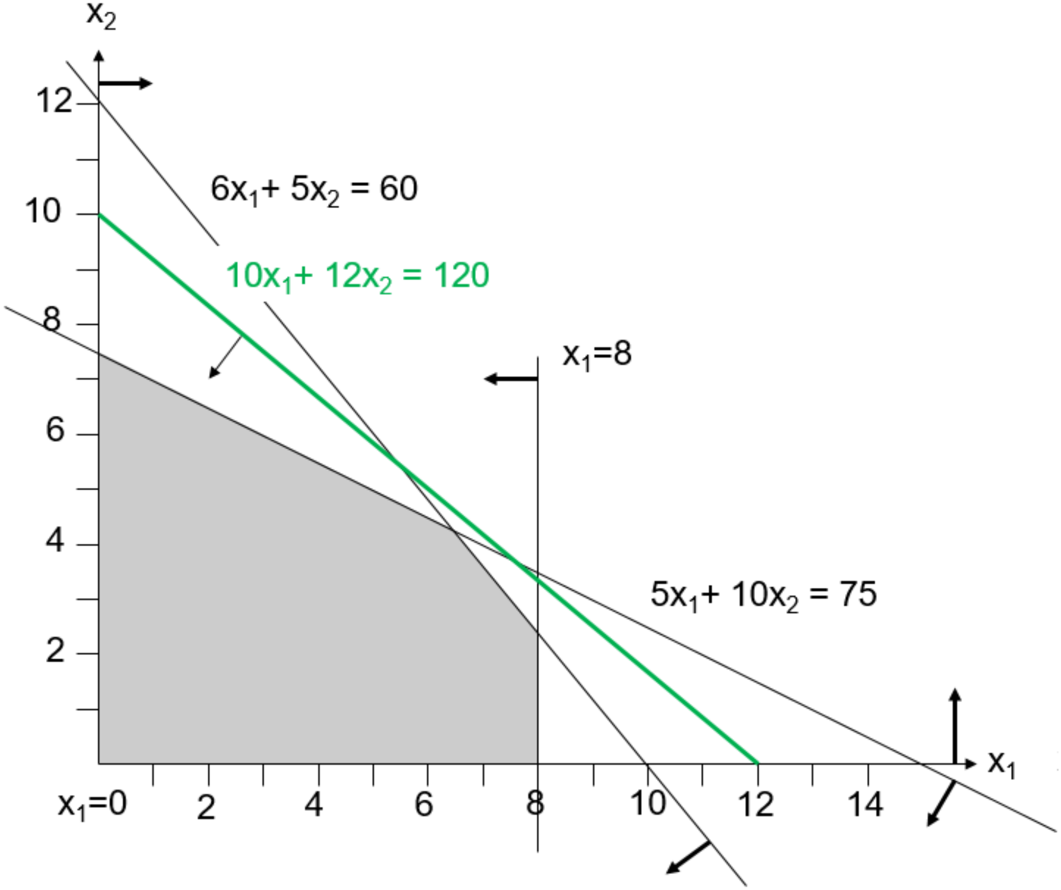
$$\text{Max } z = 5\,x_1 + 4.5\,x_2$$

s.t.

$$\begin{aligned} 6\,x_1 + 5\,x_2 &\le 60 \\ 5\,x_1 + 10\,x_2 &\le 75 \\ 1\,x_1 &\le 8 \\ 10\,x_1 + 12\,x_2 &\le 120 \\ x_1,\quad x_2 &\ge 0 \end{aligned}$$

Redundant Constraint

The slack variable of the redundant constraint is always positive and thus BV.



3.8.4 Primal Degeneracy

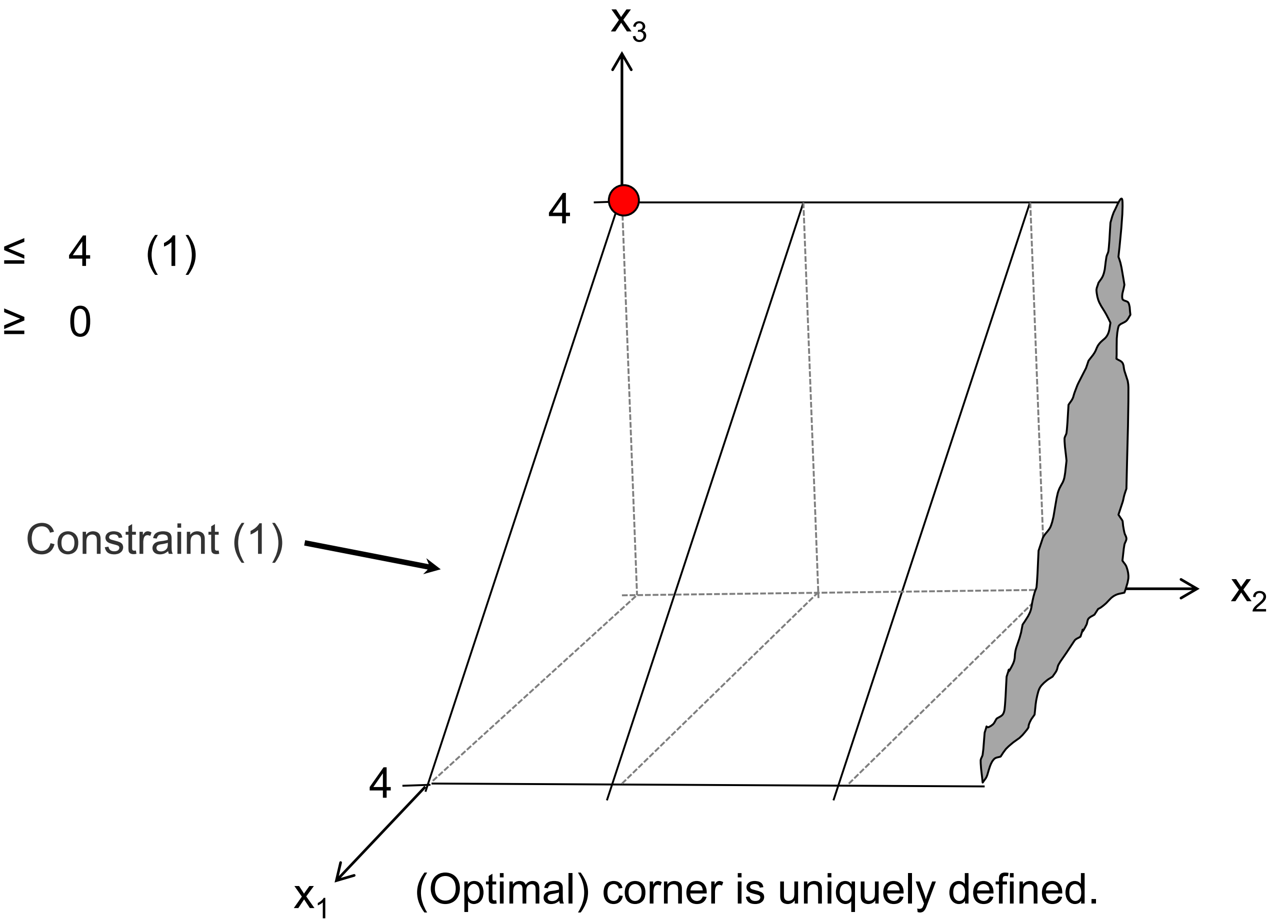
Max

$z = 0 \cdot x_1 - 1 \cdot x_2 + 1 \cdot x_3$

s.t.

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \leq 4 \quad (1)$$

$$x_1, x_2, x_3 \geq 0$$

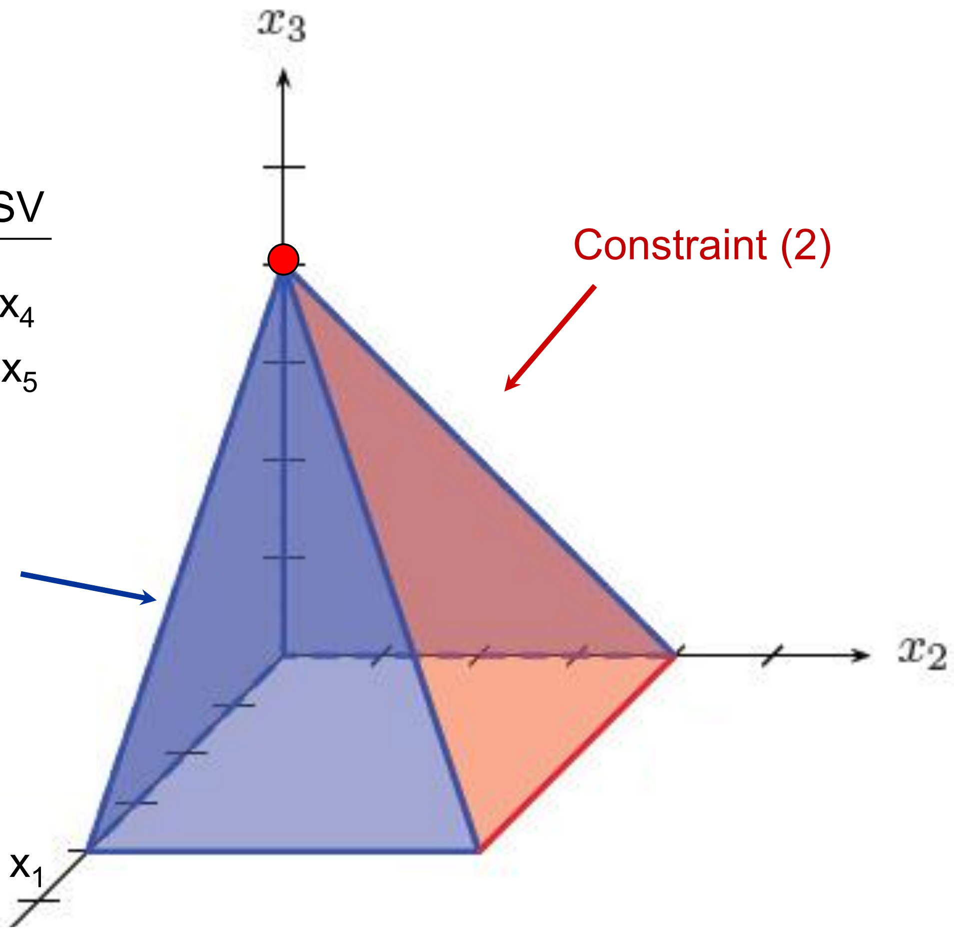


(Optimal) corner is uniquely defined.

Primal Degeneracy

Max

$$
\begin{array}{llllll}
 & 0 \cdot x_1 - 1 \cdot x_2 + 1 \cdot x_3 & = & z & & \\
\text{s.t.} & & & & & \text{SV} \\
 & 1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 & \leq & 4 & (1) & x_4 \\
 & 0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 & \leq & 4 & (2) & x_5 \\
 & x_1, x_2, x_3 & \geq & 0 & &
\end{array}
$$



Constraint (2)

Constraint (1)

Optimal tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	4	1	0	1	1	0
x_5	0	-1	1	0	-1	1
-z	-4	-1	-1	0	-1	0

Constraint (2) is not redundant. However, extreme point (0,0,4) is with Constraint (2) over-determined. x_5 is with value 0 in the basis.

Primal Degeneracy

Max $\quad -1x_2 + 1x_3$

s.t.

$$1x_1 \qquad + 1x_3 + 1x_4 \qquad = 4 \quad (1)$$

$$1x_2 + 1x_3 \qquad + 1x_5 = 4 \quad (2)$$

Pivot option 1:

Initial tableau

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	4	1	0	$\boxed{1}$	1	0
x_5	4	0	1	1	0	1
-z	0	0	-1	1	0	0

Optimal tableau

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	4	1	0	1	1	0
x_5	0	-1	1	0	-1	1
-z	-4	-1	-1	0	-1	0

Primal degeneracy: BV x_5=0

Pivot option 2:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	4	1	0	1	1	0
x_5	4	0	1	$\boxed{1}$	0	1
-z	0	0	-1	1	0	0

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	0	1	-1	0	1	-1
x_3	4	0	1	1	0	1
-z	-4	0	-2	0	0	-1

Primal degeneracy: BV x_4=0

Dual degeneracy: For NBV x_1 is $\bar{c}_1$=0

In general: For a BV with value 0, we have primal degeneracy.

3.8.5 Dual Degeneracy

Beer glass example with contributed margin of 5.4 for wheat beer glasses

$$\text{Max } z = 5.4\, x_1 + 4.5\, x_2$$

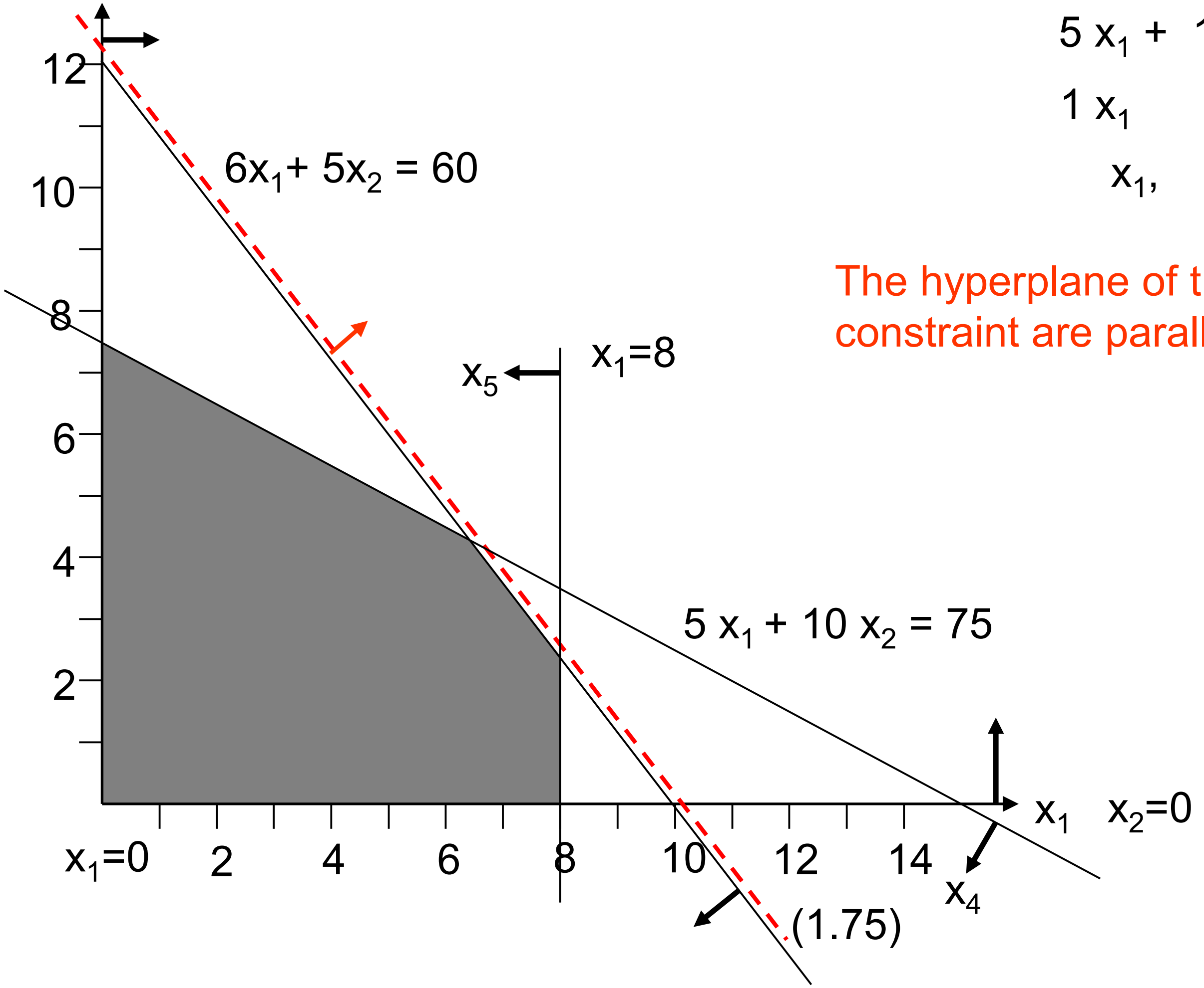
s.t.

$$\begin{aligned} 6\, x_1 + 5\, x_2 &\leq 60 \\ 5\, x_1 + 10\, x_2 &\leq 75 \\ 1\, x_1 &\leq 8 \\ x_1, \quad x_2 &\geq 0 \end{aligned}$$

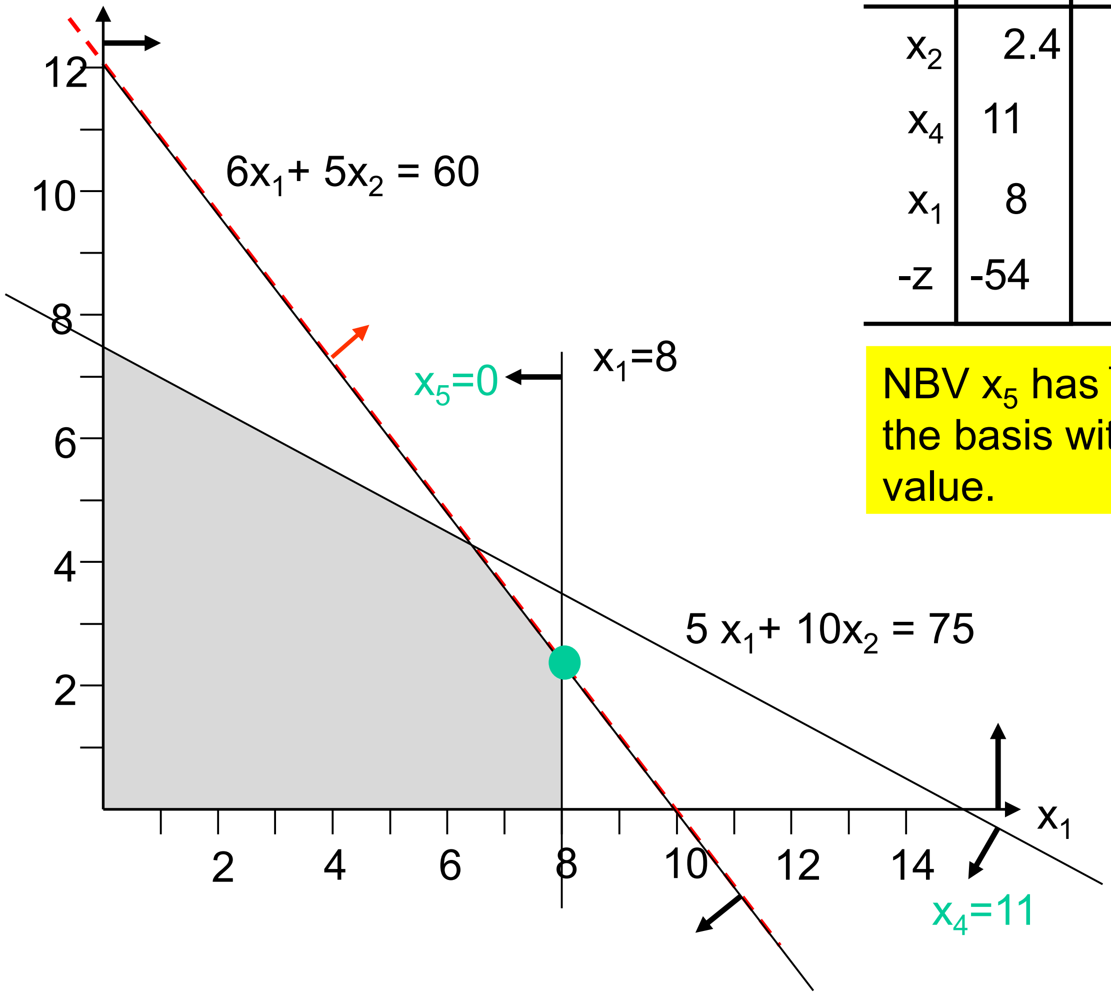
Dual Degeneracy

$$
\begin{array}{llrcr}
\text{Max } z = 5.4\,x_1 + 4.5\,x_2 & & & & \\
\text{s.t.} & & & & \\
 & 6\,x_1 + & 5\,x_2 & \leq & 60 \\
 & 5\,x_1 + & 10\,x_2 & \leq & 75 \\
 & 1\,x_1 & & \leq & 8 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

The hyperplane of the objective function and the first constraint are parallel.



Dual Degeneracy

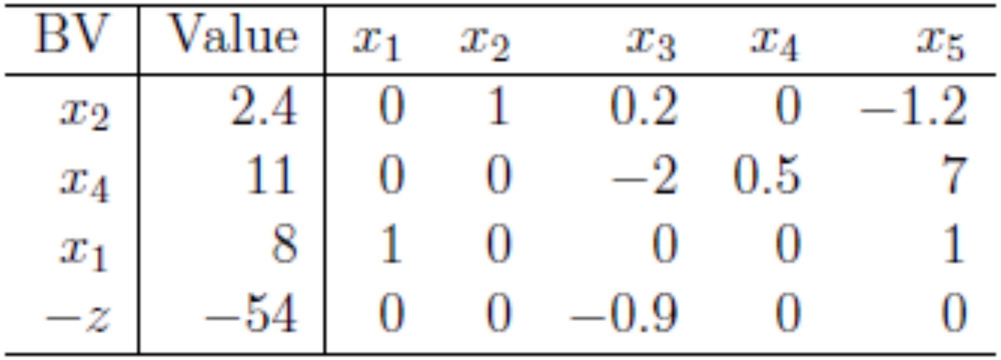


(First) optimal tableau:

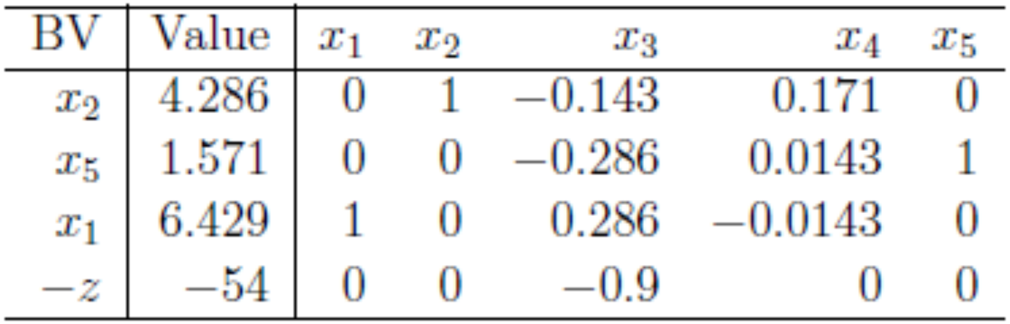
BV	Value	x_1	x_2	x_3	x_4	x_5	
x_2	2.4	0	1	0.2	0	-1.2	(1)
x_4	11	0	0	-2	0.5	7	(2)
x_1	8	1	0	0	0	1	(3)
-z	-54	0	0	-0.9	0	0	(4)

NBV x_5 has $\bar{c}_5=0$ and can therefore be moved into the basis without decreasing the objective function value.

Dual Degeneracy



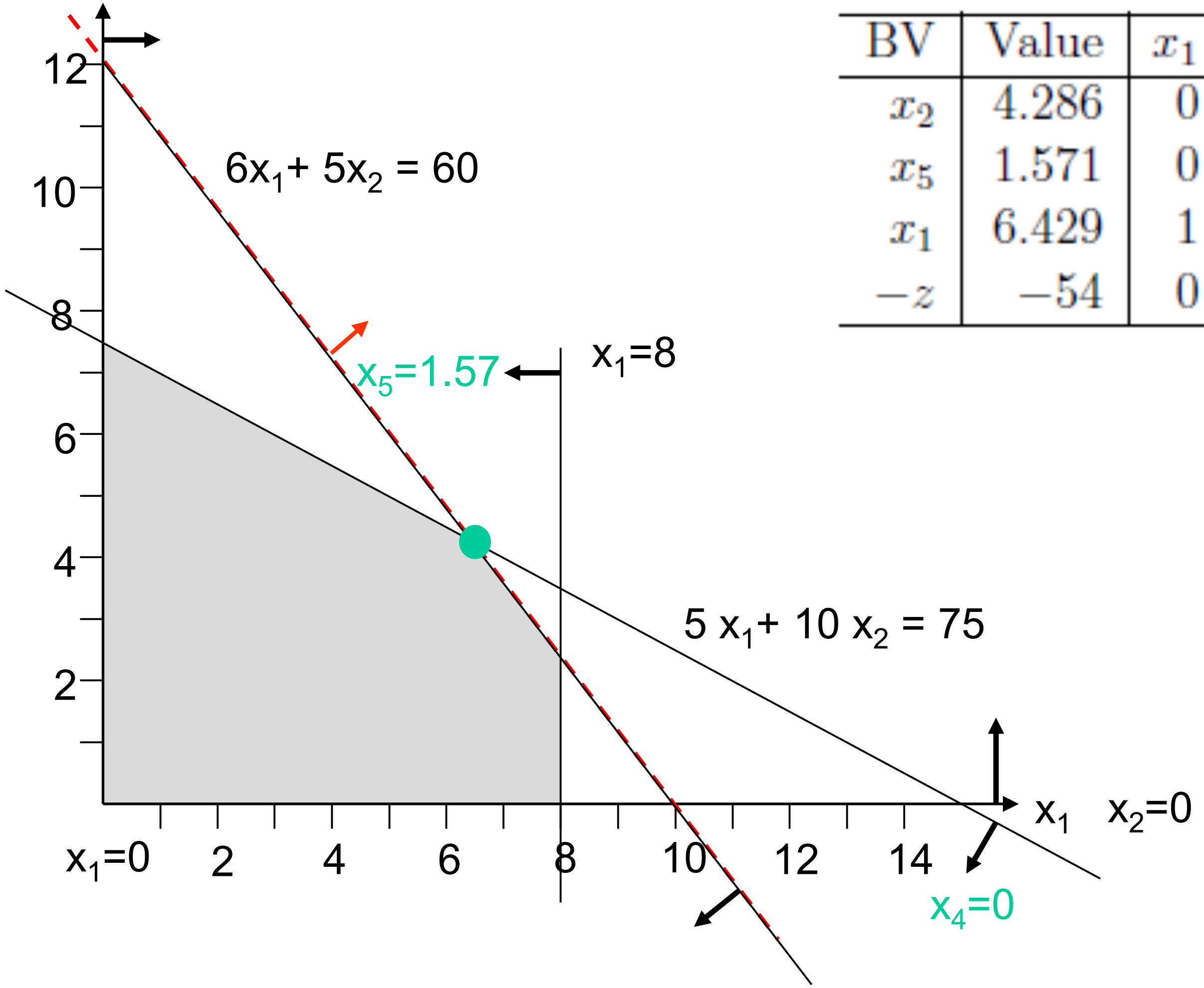
BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	2.4	0	1	0.2	0	-1.2
x_4	11	0	0	-2	0.5	7
x_1	8	1	0	0	0	1
$-z$	-54	0	0	-0.9	0	0



BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4.286	0	1	-0.143	0.171	0
x_5	1.571	0	0	-0.286	0.0143	1
x_1	6.429	1	0	0.286	-0.0143	0
$-z$	-54	0	0	-0.9	0	0

→ Change of the basis, no change in objective function value

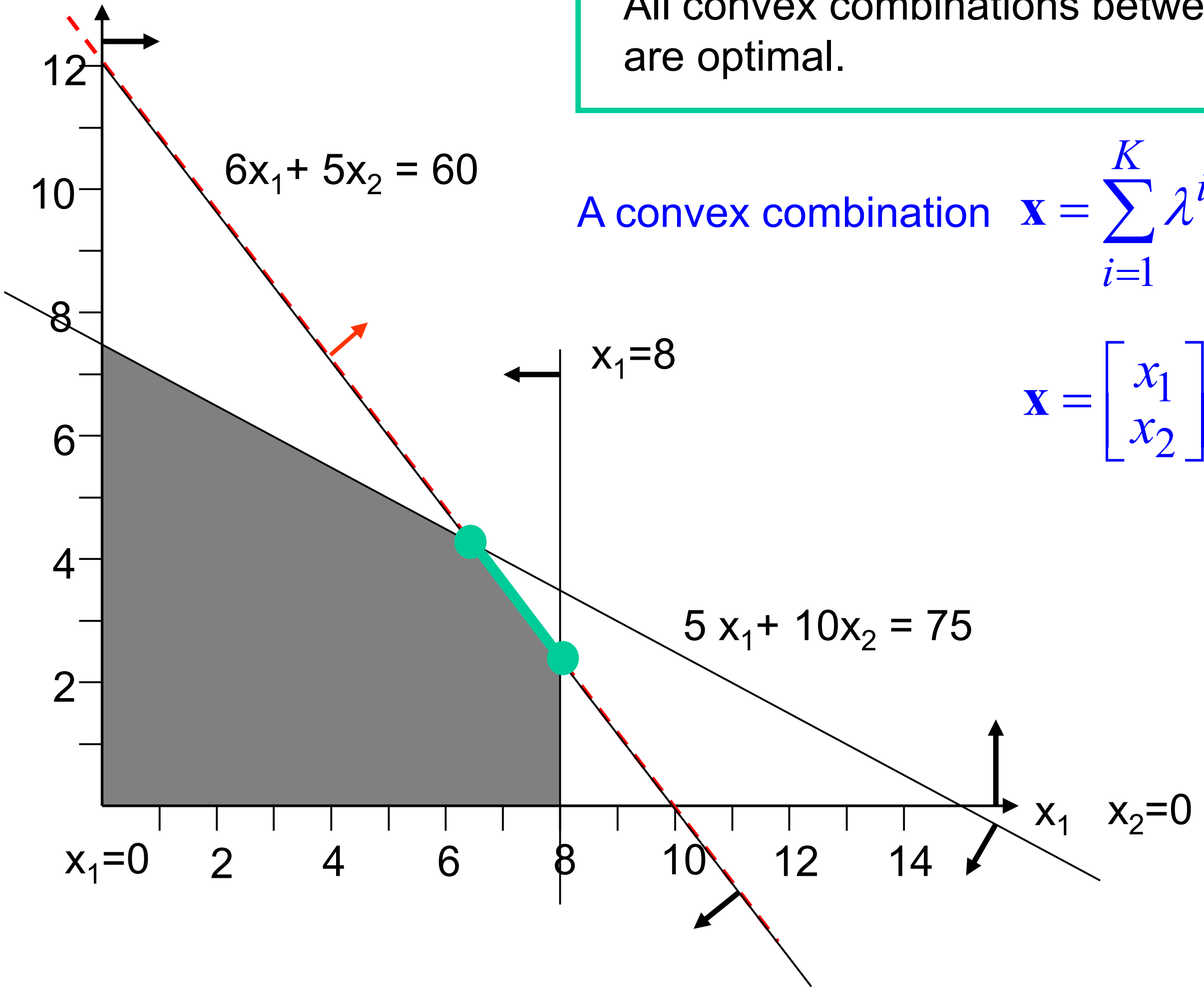
Dual Degeneracy



(Second) optimal tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4.286	0	1	−0.143	0.171	0
x_5	1.571	0	0	−0.286	0.0143	1
x_1	6.429	1	0	0.286	−0.0143	0
$-z$	−54	0	0	−0.9	0	0

Dual Degeneracy



All convex combinations between the two optimal solutions are optimal.

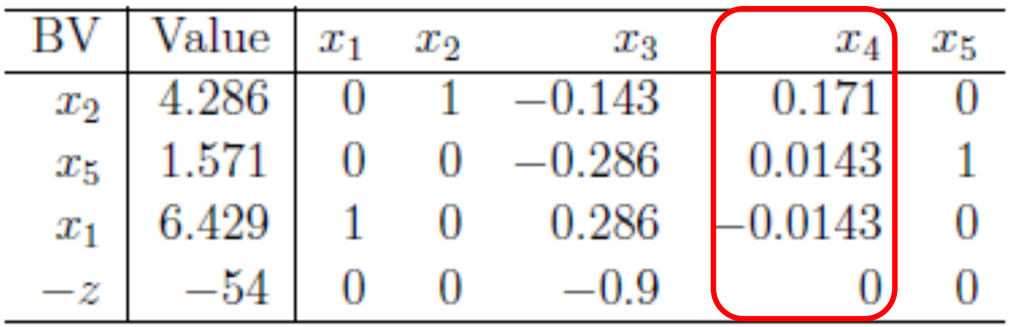
A convex combination $\mathbf{x} = \sum_{i=1}^{K} \lambda^i \mathbf{x}^i, \quad \sum_{i=1}^{K} \lambda^i = 1, \lambda^i \geq 0$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda^1 \begin{bmatrix} 8 \\ 2.4 \end{bmatrix} + \lambda^2 \begin{bmatrix} 6.42 \\ 4.28 \end{bmatrix}$$

$$\lambda^1 + \lambda^2 = 1$$

$$\lambda^1, \quad \lambda^2 \geq 0$$

Dual Degeneracy



BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4.286	0	1	−0.143	0.171	0
x_5	1.571	0	0	−0.286	0.0143	1
x_1	6.429	1	0	0.286	−0.0143	0
$-z$	−54	0	0	−0.9	0	0

<u>How can we see it in the tableau</u>:
At least one NBV with $\bar{c}$=0.

3.9 Sensitivity Analysis

Motivation and Assumptions

Motivation

- So far we assumed that all data is deterministic.
- However, often data is not exactly known.
- Sensitivity analysis explores how the change of one data impacts the optimal solution.

Assumption

We always change one data at a time, leaving all other data as it is (ceteris paribus).

3.9.1

Changing the Objective Function Coefficient of a Non-Basic Variable

Changing the OF-Coefficient of a NBV

Initial tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	60	6	5	8	1	0	0
x_5	75	5	10	5	0	1	0
x_6	8	1	0	0	0	0	1
$-z$	0	5	4.5	$\Delta c_3 +$ 6	0	0	0

Optimal tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	4.28	0	1	−0.28	−0.14	0.17	0
x_6	1.57	0	0	−1.57	−0.28	0.14	1
x_1	6.42	1	0	1.57	0.28	−0.14	0
$-z$	−51.42	0	0	$\Delta c_3 + -0.57$	−0.78	−0.05	0

$\Rightarrow \Delta \bar{c}_3^* < 0.57$

for no change in
optimal solution

3.9.2

Changing the Objective Function Coefficient of a Basic Variable

Motivation and Assumption

- How much can we change the objective function coefficient of a basic variable without changing the basis?
- Note: the change of the coefficient will lead to a new optimal objective function value. Also the value of the basic variables changes. However, the basis does not change.

Change of the OF-Coefficient of a BV

Initial tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	60	6	5	8	1	0	0
x_5	75	5	10	5	0	1	0
x_6	8	1	0	0	0	0	1
$-z$	0	5	4.5	6	0	0	0

Optimal tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	4.28	0	1	-0.28	-0.14	0.17	0
x_6	1.57	0	0	-1.57	-0.28	0.14	1
x_1	6.42	1	0	1.57	0.28	-0.14	0
$-z$	-51.42	0	0	-0.57	-0.78	-0.05	0

Optimal tableau in standard form:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	4.28	0	1	-0.28	-0.14	0.17	0
x_6	1.57	0	0	-1.57	-0.28	0.14	1
x_1	6.42	1	0	1.57	0.28	-0.14	0
$-z$	$-51.42 - 6.42\Delta c_1$	0	0	$-0.57 - 1.57\Delta c_1$	$-0.78 - 0.28\Delta c_1$	$-0.05 + 0.14\Delta c_1$	0

Change of the OF-Coefficient of a BV

Optimal tableau in standard form:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	4.28	0	1	-0.28	-0.14	0.17	0
x_6	1.57	0	0	-1.57	-0.28	0.14	1
x_1	6.42	1	0	1.57	0.28	-0.14	0
$-z$	$-51.42 - 6.42\Delta c_1$	0	0	$-0.57 - 1.57\Delta c_1$	$-0.78 - 0.28\Delta c_1$	$-0.05 + 0.14\Delta c_1$	0

For optimality: $\bar{c}_j \leq 0$

$$\Rightarrow -0.57 - 1.57\Delta c_1 \leq 0 \Rightarrow \Delta c_1 \geq -\frac{0.57}{1.57} = -0.369$$

$$-0.78 - 0.28\Delta c_1 \leq 0 \Rightarrow \Delta c_1 \geq -2.785$$

$$-0.05 + 0.14\Delta c_1 \leq 0 \Rightarrow \Delta c_1 \leq 0.35$$

$\Rightarrow \max\{\underline{-0.369}, -2.785\} \leq \Delta c_1 \leq 0.35$ ⇒ c_1 can change by up to $\underline{\underline{0.35}}$ before changing the basis