

Solutions

1 Decision Theory

1.1 Decision Context

1.1.1 Basics

Question 1

- Level of uncertainty
 - Certainty
 - Uncertainty
 - Risk
- Number of objectives
 - One objective
 - Multiple objectives
- Number of decision makers
 - Single person decision
 - Group decision
- Time line
 - Static
 - Dynamic
- Behavior of opponent
 - Random
 - Strategic

Question 2

a) Alternative:

A possibility for the small investor to invest his savings.

The set of alternatives:

Set of all possibilities for the small investor to invest his savings.

States of nature (scenarios):

Positive (negative) development of the stock market.

b)

a_1 : no investment

a_2 : investment of 3,000 Euro into fixed-income securities.

a_3 : investment of 6,000 Euro into fixed income securities.

a_4 : investment of 4,000 Euro into equity mutual fund.

a_5 : investment of 8,000 Euro into equity mutual fund.

a_6 : investment of 3,000 Euro into fixed-income securities and 4,000 Euro into equity mutual fund.

c)

	s_1	s_2
a_1	8,000	8,000
a_2	8,300	8,300
a_3	8,600	8,600
a_4	8,800	7,200
a_5	9,600	6,400
a_6	9,100	7,500

1.1.2 Efficient Alternatives

Question 3

	a_1	a_2	a_1	a_3	a_1	a_4	a_2	a_3	a_2	a_4	a_3	a_4	
s_1	13	<	14	13	>	11	13	>	12	14	>	12	
s_2	21	>	15	21	>	18	21	>	19	15	<	18	
s_3	\vdots	\vdots	25	>	24	25	\geq	25	\vdots	\vdots	20	<	25
s_4			18	>	13	18	>	13			14	>	13
s_5			10	<	13	10	?	x			18	?	x

Neither alternative a_1 nor alternative a_2 dominate the other one respectively.

Neither alternative a_1 nor alternative a_3 dominate the other one respectively.

Alternative a_1 dominates alternative a_4 if: $x \leq 10$

Neither alternative a_2 nor alternative a_3 dominate the other one respectively.

Neither alternative a_2 nor alternative a_4 dominate the other one respectively no matter the value of x .

Alternative a_4 dominates alternative a_3 if: $x \geq 13$

Result: All alternatives are efficient if $x \in \{11,12\}$

Question 4

	s_1	s_2	s_3	s_4	s_5	s_6
a_1	-1	0	2	1	3	2
a_2	alternative inefficient					
a_3	alternative inefficient					
a_4	1	0	2	-1	4	2
a_5	0	1	2	1	3	-2
a_6	alternative inefficient					

Alternative a_2 is dominated by alternative a_1 , a_4 and a_6 .

Alternative a_3 is dominated by alternative a_5 .

Alternative a_6 is dominated by alternative a_1 and a_4 .

1.2 Decision under Uncertainty

1.2.1 Decision rules

Question 5

Criteria	Value
Level of uncertainty	uncertainty
Number of objectives	1
Number of decision makers	1
Timeline	static
Behavior of opponent	random

Question 6

(1) Maximax-Rule

- Decision maker assumes, that the for the alternative chosen the best scenario will occur respectively.
- Alternative with the highest maximum profit is chosen.
- Rule for a risk seeking decision maker.

(1) Maximin-Rule

- Decision maker assumes, that the for the alternative chosen the worst scenario will occur respectively.
- Alternative with highest „guaranteed“ minimum profit is chosen.
- Rule for risk averse decision maker.

(3) Laplace-Rule

- Decision makers assumes that all scenarios occur with equal probability.
- Alternative with maximum average profit is chosen.
- Rule for risk neutral decision maker.

(4) Minimax-Regret-Rule

- The decision maker determines the maximum possible profit for every scenario.
- The decision maker calculates how much profit he would lose, if he had not chosen the best alternative for this scenario (=Regret).
- Alternative with lowest maximum regret is chosen.
- Rule for risk averse decision maker.

(4) Hurwicz-Rule

- Alternative which maximizes the linear combination of the maximum and minimum profit.
- for $\lambda = 1$ the rule equals the Maximax-Rule (risk seeking decision maker)
- for $\lambda = 0$ the rule equals the Maximin-Rule (risk averse decision maker)

Question 7

a) All alternatives are efficient.

b)

	s_1	s_2	s_3	s_4	$\max_{j=1,2,3,4} \{e_{ij}\}$	$\min_{j=1,2,3,4} \{e_{ij}\}$	$\frac{1}{4} \sum_{j=1,2,3,4} e_{ij}$
a_1	-1	5	0	3	5	-1 ⁽²⁾	1.75
a_2	4	-2	1	2	4	-2	1.25
a_3	3	7	1	-3	7 ⁽¹⁾	-3	2.00 ⁽³⁾
a_4	1	-1	0	0	1	-1 ⁽²⁾	0.00
$\max_{i=1,2,3,4} \{e_{ij}\}$	4	7	1	3			

Regret-Matrix

	s_1	s_2	s_3	s_4	$\max_{j=1,2,3,4} \{r_{ij}\}$
a_1	5	2	1	0	5 ⁽⁴⁾
a_2	0	9	0	1	9
a_3	1	0	0	6	6
a_4	3	8	1	3	8

Hurwicz-Rule

$$\lambda_1 = 0.25:$$

$$a_1 : 0.25 \times 5 + (1 - 0.25) \times (-1) = 0.5^{(5)}$$

$$a_2 : 0.25 \times 4 + (1 - 0.25) \times (-2) = -0.5$$

$$a_3 : 0.25 \times 7 + (1 - 0.25) \times (-3) = -0.5$$

$$a_4 : 0.25 \times 1 + (1 - 0.25) \times (-1) = -0.5$$

$$\lambda_2 = 0.75:$$

$$a_1 : 0.75 \times 5 + (1 - 0.75) \times (-1) = 3.5$$

$$a_2 : 0.75 \times 4 + (1 - 0.75) \times (-2) = 2.5$$

$$a_3 : 0.75 \times 7 + (1 - 0.75) \times (-3) = 4.5^{(5)}$$

$$a_4 : 0.75 \times 1 + (1 - 0.75) \times (-1) = 0.5$$

- (1) Decision if Maximax-Rule is used
- (2) Decision if Maximin-Rule is used
- (3) Decision if Laplace-Rule is used
- (4) Decision if Minimax-Regret-Rule is used
- (5) Decision if Hurwicz-Rule is used

Question 8

- a)
- a_1 : Purchase of 0 nights.
 - a_2 : Purchase of 1,000 nights.
 - a_3 : Purchase of 2,000 nights.
 - a_4 : Purchase of 3,000 nights.

- s_1 : Demand of 3,000 nights.
- s_2 : Demand of 1,800 nights.
- s_3 : Demand of 900 nights.

	s_1	s_2	s_3
a_1	0	0	0
a_2	10,000	10,000	0
a_3	20,000	0	-90,000
a_4	30,000	-90,000	-180,000

- b)

	s_1	s_2	s_3
a_1	alternative inefficient		
a_2	10,000	10,000	0
a_3	20,000	0	-90,000
a_4	30,000	-90,000	-180,000

Reason: alternative a_1 is dominated by a_2 .

c)

	s_1	s_2	s_3	$\max_{j=1,2,3} \{e_{ij}\}$	$\min_{j=1,2,3} \{e_{ij}\}$	$\frac{1}{3} \sum_{j=1,2,3} e_{ij}$
a_2	10,000	10,000	0	10,000	$0^{(2)}$	$6,666,\bar{6}^{(3)}$
a_3	20,000	0	-90,000	20,000	-90,000	$-23,333,\bar{3}$
a_4	30,000	-90,000	-180,000	$30,000^{(1)}$	-180,000	-80,000
$\max_{i=2,3,4} \{e_{ij}\}$	30,000	10,000	0			

Regret-Matrix

	s_1	s_2	s_3	$\max_{j=1,2,3} \{r_{ij}\}$
a_2	20,000	0	0	$20,000^{(4)}$
a_3	10,000	10,000	90,000	90,000
a_4	0	100,000	180,000	180,000

- (1) Decision if Maximax-Rule is used
 (2) Decision if Maximin-Rule is used
 (3) Decision if Laplace-Rule is used
 (4) Decision if Minimax-Regret-Rule is used

Question 9

- a) a_1 : She buys no bunch.
 a_2 : She buys one bunch.
 a_3 : She buys two bunches.
 a_4 : She buys three bunches.
 a_5 : She buys four bunches.

- s_1 : She sells no bunch.
 s_2 : She sells one bunch.
 s_3 : She sells two bunches.
 s_4 : She sells three bunches.
 s_5 : She sells four bunches.

	s_1	s_2	s_3	s_4	s_5
a_1	0	0	0	0	0
a_2	-6	2	2	2	2
a_3	-12	-4	4	4	4
a_4	-18	-10	-2	6	6
a_5	-24	-16	-8	0	8

b) All alternatives are efficient.

c)

	s_1	s_2	s_3	s_4	s_5	$\max_{j=1,\dots,5} \{e_{ij}\}$	$\min_{j=1,\dots,5} \{e_{ij}\}$	$\frac{1}{5} \sum_{j=1,\dots,5} e_{ij}$
a_1	0	0	0	0	0	0	$0^{(2)}$	0.0
a_2	-6	2	2	2	2	2	-6	$0.4^{(3)}$
a_3	-12	-4	4	4	4	4	-12	-0.8
a_4	-18	-10	-2	6	6	6	-18	-3.6
a_5	-24	-16	-8	0	8	$8^{(1)}$	-24	-8.0
$\max_{i=1,\dots,5} \{e_{ij}\}$	0	2	4	6	8			

Regret-Matrix

	s_1	s_2	s_3	s_4	s_5	$\max_{j=1,\dots,5} \{r_{ij}\}$
a_1	0	2	4	6	8	8
a_2	6	0	2	4	6	$6^{(4)}$
a_3	12	6	0	2	4	12
a_4	18	12	6	0	2	18
a_5	24	18	12	6	0	24

(1) Decision if Maximax-Rule is used

(2) Decision if Maximin-Rule is used

(3) Decision if Laplace-Rule is used

(4) Decision if Minimax-Regret-Rule is used

Question 10

	s_1	s_2	s_3	s_4	s_5
a_1	100,000	100,000	150,000	250,000	300,000
a_2	50,000	200,000	100,000	250,000	300,000
a_3	200,000	150,000	150,000	200,000	200,000
$\min_{i=1,\dots,3} \{e_{ij}\}$	50,000	100,000	100,000	200,000	200,000

Regret-Matrix

	s_1	s_2	s_3	s_4	s_5	$\max_{j=1,\dots,5} \{r_{ij}\}$
a_1	50,000	0	50,000	50,000	100,000	100,000 ⁽¹⁾
a_2	0	100,000	0	50,000	100,000	100,000 ⁽¹⁾
a_3	150,000	50,000	50,000	0	0	150,000

⁽¹⁾ Decision if Minimax-Regret-Rule is used

1.3 Decision under Risk

1.3.1 Expected value criterion

Question 11

Criteria	Value
Level of uncertainty	risk
Number of objectives	1
Number of decision makers	1 risk neutral decision maker
Timeline	static
Behavior of opponent	random

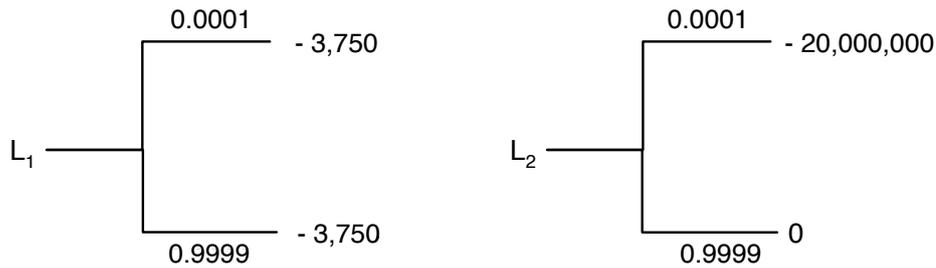
Question 12

- a) a_1 : Taking out insurance.
 a_2 : Not taking out insurance.

s_1 : Paintings are stolen.
 s_2 : Paintings are not stolen.

	$p_1 = 0.0001$	$p_2 = 0.9999$
	s_1	s_2
a_1	-3,750	-3,750
a_2	-20,000,000	0

b)



c)

$$\begin{aligned} EV(L_1) &= 0.0001 \cdot (-3,750) + 0.9999 \cdot (-3,750) \\ &= -3,750 \end{aligned}$$

$$\begin{aligned} EV(L_2) &= 0.0001 \cdot (-20,000,000) + 0.9999 \cdot 0 \\ &= -2,000 \end{aligned}$$

The art collector decides will not take out the insurance. Reason: $EV(L_2) > EV(L_1)$

1.3.2 Expected utility criterion

Question 13

Criteria	Value
Level of uncertainty	risk
Number of objectives	1
Number of decision makers	1 decision maker with any risk attitude
Timeline	static
Behavior of opponent	random

Question 14

Complete order

From the point of the decision maker for two results e_1 and e_2 it must hold that $e_1 \succ e_2$ or $e_2 \succ e_1$ or $e_1 \sim e_2$.

Continuity

If it holds that $e_1 \succ e_2 \succ e_3$ for the decision maker, then there exists a p with $0 < p < 1$ for which it holds that $L_1 = (1, e_2) \sim L_2 = (p, e_1; 1 - p, e_3)$.

Independence

If it holds that $e_1 \sim e_2$ for the decision maker, then it has to hold for e_3 and p with $0 < p < 1$: $L_1 = (p, e_1; 1 - p, e_3) \sim L_2 = (p, e_2; 1 - p, e_3)$

Compound lotteries

The decision maker is indifferent between a compound and non-compound lottery, if the probabilities and results do not change.

Unequal probabilities

If it holds that $e_1 \succ e_2$ for the decision maker, then it must hold for $p_1 > p_2$: $L_1 = (p_1, e_1; 1 - p_1, e_2) \succ L_2 = (p_2, e_1; 1 - p_2, e_2)$

Question 15

a)

$$\begin{aligned}EV(L) &= 0.80 \cdot 10 + 0.20 \cdot 3 \\ &= 8.6\end{aligned}$$

$$\begin{aligned}EU(L) &= 0.80 \cdot u(10) + 0.20 \cdot u(3) \\ &= 0.80 \cdot 0.75 + 0.20 \cdot 0.5225 \\ &= 0.7045\end{aligned}$$

$$\begin{aligned}CE(L) &= u^{-1}(0.7045) \\ &= 20 \cdot \sqrt{0.7045 - 0.5} \\ &= 9.044\end{aligned}$$

$$\begin{aligned}RP(L) &= EV(L) - CE(L) \\ &= 8.6 - 9.044 \\ &= -0.444\end{aligned}$$

b) Mr. Hinz is risk seeking because he would pay 0.444 Euro more for the participation in the lottery than a risk neutral decision maker, it holds $RP < 0$.

Question 16

a)

$$\begin{aligned}EV(L) &= 0.64 \cdot 10 + 0.36 \cdot 0 \\ &= 6.40\end{aligned}$$

$$\begin{aligned}EU(L) &= 0.64 \cdot u(10) + 0.36 \cdot u(0) \\ &= 0.64 \cdot 1 + 0.36 \cdot 0 \\ &= 0.64\end{aligned}$$

$$\begin{aligned}CE(L) &= u^{-1}(0.64) \\ &= 10 \cdot 0.64^2 \\ &= 4.096\end{aligned}$$

$$\begin{aligned}RP(L) &= EV(L) - CE(L) \\ &= 6.40 - 4.096 \\ &= 2.304\end{aligned}$$

b) Mr. Kunz is risk averse, because he would pay 2.304 Euro less for the participation in the lottery than a risk neutral decision maker, it holds $RP > 0$.

- c) Mr. Kunz would pay a maximum of 4.096 Euro and a risk neutral decision maker a maximum of 6.40 Euro.

Question 17

a)

$$\begin{aligned} EV(L) &= 0.25 \cdot 100 + 0.75 \cdot 0 \\ &= 25 \end{aligned}$$

$$\begin{aligned} EU(L) &= 0.25 \cdot u(100) + 0.75 \cdot u(0) \\ &= 0.25 \cdot 1 + 0.75 \cdot 0 \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} CE(L) &= u^{-1}(0.25) \\ &= 100 \cdot \sqrt{0.25} \\ &= 50 \end{aligned}$$

$$\begin{aligned} RP(L) &= EV(L) - CE(L) \\ &= 25 - 50 \\ &= -25 \end{aligned}$$

- b) The decision maker is risk seeking, because he would pay 25 Euro more for the participation in the lottery than a risk neutral decision maker, it holds $RP < 0$.

c)

$$\begin{aligned} 0.25 &\stackrel{!}{=} p \cdot u(80) + (1 - p) \cdot u(20) \\ 0.25 &= p \cdot 0.64 + (1 - p) \cdot 0.04 \\ 0.25 &= p \cdot 0.60 + 0.04 \\ p &= 0.35 \end{aligned}$$

Question 18

a)

$$\begin{aligned} EV(L) &= 0.20 \cdot 100 + 0.50 \cdot 50 + 0.30 \cdot 0 \\ &= 45 \end{aligned}$$

$$CE(L) = 50$$

$$\begin{aligned} RP(L) &= EV(L) - CE(L) \\ &= 45 - 50 \\ &= -5 \end{aligned}$$

The decision maker is risk seeking, because he would pay 5 Euro more for the participation in the lottery than a risk neutral decision maker, it holds $RP < 0$.

b)

$$\begin{aligned}EU(L) &= 0.20 \cdot u(100) + 0.50 \cdot u(50) + 0.30 \cdot u(0) \\ &= 0.20 \cdot 1 + 0.50 \cdot 0.707 + 0.30 \cdot 0 \\ &= 0.55\end{aligned}$$

$$0.55 \stackrel{!}{=} 0.50 \cdot u(y) + 0.50 \cdot u\left(\frac{y}{4}\right)$$

$$0.55 = 0.50 \cdot \frac{1}{10} \cdot \sqrt{y} + 0.50 \cdot \frac{1}{10} \cdot \sqrt{\frac{y}{4}}$$

$$11 = \sqrt{y} + \frac{1}{2} \cdot \sqrt{y}$$

$$22 = 3\sqrt{y}$$

$$y = 53.78$$

Question 19

$$\begin{aligned}EU(L_1) &= 0.5 \cdot u(20,000) + 0.5 \cdot u(40,000) \\ &= 0.5 \cdot 0.12 + 0.5 \cdot 0.28 \\ &= 0.2\end{aligned}$$

$$EU(L_1) \stackrel{!}{=} EU(L_2)$$

$$0.2 = 0.7 \cdot u(y) + 0.3 \cdot u(0)$$

$$0.2 = 0.7 \cdot \left[\frac{1}{2} \cdot \left(\frac{y}{100,000} \right)^2 + \frac{y}{200,000} \right]$$

$$0.5714 = \frac{y^2}{100,000^2} + \frac{y}{100,000}$$

$$0 = \frac{y^2}{100,000^2} + \frac{y}{100,000} - 0.5714$$

$$y = 40,631.15$$

Question 20

a)

$$\begin{aligned}EV(L_1) &= 1 \cdot 240,000 \\ &= 240,000\end{aligned}$$

$$\begin{aligned}EV(L_2) &= 0.3 \cdot 450,000 + 0.7 \cdot 150,000 \\ &= 240,000\end{aligned}$$

b)

$$\begin{aligned}
 EU(L_1) &= 1 \cdot u(240,000) \\
 &= 1 \cdot \frac{1}{1,000} \cdot \sqrt{240,000} \\
 &= 0.49
 \end{aligned}$$

$$\begin{aligned}
 EU(L_2) &= 0.3 \cdot u(450,000) + 0.7 \cdot u(150,000) \\
 &= 0.3 \cdot \left(\frac{1}{2} + \left(\frac{450,000 - 250,000}{396,850} \right)^{\frac{3}{2}} \right) + 0.7 \cdot \frac{1}{1,000} \cdot \sqrt{150,000} \\
 &= 0.3 \cdot 0.86 + 0.7 \cdot 0.39 \\
 &= 0.53
 \end{aligned}$$

He would choose the performance-based contract. Reason: $EU(L_2) > EU(L_1)$

c)

$$\begin{aligned}
 CE(L_2) &\stackrel{!}{=} u^{-1}(0.53) \\
 &= \left(0.53 - \frac{1}{2} \right)^{\frac{2}{3}} \cdot 396,850 + 250,000 \\
 &= 288,315
 \end{aligned}$$

The safe salary would have to be more than 288,315 Euro.

Question 21

a) Arrow-Pratt measure subject to a :

$$\begin{aligned}
 AP(x) &= -\frac{u''(x)}{u'(x)} \\
 &= -\frac{a \cdot (a-1) \cdot x^{(a-2)}}{a \cdot x^{(a-1)}} \\
 &= \frac{1-a}{x}
 \end{aligned}$$

Risk attitude	Arrow-Pratt measure	Parameter a
Risk neutral	$AP(x) = 0$	$a = 1$
Risk seeking	$AP(x) < 0$	$a > 1$
Risk averse	$AP(x) > 0$	$a < 1$

b) (1)

$$\begin{aligned}
 EU(L) &= 0.5 \cdot u(1) + 0.5 \cdot u(0) \\
 &= 0.5 \cdot 1^a + 0.5 \cdot 0^a \\
 &= 0.5 \cdot 1 + 0.5 \cdot 0 \\
 &= 0.5
 \end{aligned}$$

$$\begin{aligned}
 CE(L) &\stackrel{!}{=} u^{-1}(0.5) \\
 &= 0.5^{\frac{1}{a}}
 \end{aligned}$$

(2)

$$\begin{aligned} CE(L) &= 0.5^{\frac{4}{3}} \\ &= 0.3968 \end{aligned}$$

He would participate in the lottery, because the payment of 0.30 Euro is smaller than 0.3968 Euro (= Maximum amount he would pay).

Question 22

a) Mr. Roland is risk neutral, because it holds:

$$AP(x) = -\frac{u''(x)}{u'(x)} = -\frac{0}{\underbrace{\frac{1}{100,000}}_{=0}}$$

Mr. Berger is risk seeking in the interval $0 \leq x \leq 50,000$, because it holds:

$$\begin{aligned} AP(x) &= -\frac{v''(x)}{v'(x)} \\ &= -\frac{\frac{1}{2,500,000,000}}{\frac{x}{2,500,000,000}} \\ &= -\frac{1}{\underbrace{x}_{<0}} \end{aligned}$$

Mr. Berger is risk averse in the interval $50,000 \leq x \leq 100,000$, because it holds:

$$\begin{aligned} AP(x) &= -\frac{v''(x)}{v'(x)} \\ &= -\frac{\frac{-1}{4 \cdot 200,000^2 \cdot \left(\frac{x-50,000}{200,000}\right)^{\frac{3}{2}}}}{\frac{1}{2 \cdot 200,000 \cdot \left(\frac{x-50,000}{200,000}\right)^{\frac{1}{2}}}} \\ &= \frac{1}{400,000 \cdot \left(\frac{x-50,000}{200,000}\right)} \\ &= \frac{1}{\underbrace{2 \cdot (x-50,000)}_{>0}} \end{aligned}$$

b) Mr. Roland is indifferent between both products, because it holds:

$$\begin{aligned} EU(L_1) &= 0.3 \cdot u(50,000) + 0.5 \cdot u(90,000) + 0.2 \cdot u(100,000) \\ &= 0.3 \cdot 0.5 + 0.5 \cdot 0.9 + 0.2 \cdot 1.0 \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} EU(L_2) &= 1.0 \cdot u(80,000) \\ &= 1.0 \cdot 0.8 \\ &= 0.8 \end{aligned}$$

Mr. Berger favors product B, because it holds:

$$\begin{aligned}EU(L_1) &= 0.3 \cdot v(50,000) + 0.5 \cdot v(90,000) + 0.2 \cdot v(100,000) \\&= 0.3 \cdot 0.5 + 0.5 \cdot 0.95 + 0.2 \cdot 1.0 \\&= 0.825\end{aligned}$$

$$\begin{aligned}EU(L_2) &= 1.0 \cdot v(80,000) \\&= 1.0 \cdot 0.89 \\&= 0.89\end{aligned}$$

c) Mr. Roland is still indifferent between both products, because it holds:

$$\begin{aligned}EU(L_1) &= 0.3 \cdot u(0) + 0.5 \cdot u(40,000) + 0.2 \cdot u(50,000) \\&= 0.3 \cdot 0 + 0.5 \cdot 0.4 + 0.2 \cdot 0.5 \\&= 0.3\end{aligned}$$

$$\begin{aligned}EU(L_2) &= 1.0 \cdot u(30,000) \\&= 1.0 \cdot 0.3 \\&= 0.3\end{aligned}$$

Mr. Berger now favors product A, because it holds:

$$\begin{aligned}EU(L_1) &= 0.3 \cdot v(0) + 0.5 \cdot v(40,000) + 0.2 \cdot v(50,000) \\&= 0.3 \cdot 0 + 0.5 \cdot 0.32 + 0.2 \cdot 0.5 \\&= 0.26\end{aligned}$$

$$\begin{aligned}EU(L_2) &= 1.0 \cdot v(30,000) \\&= 1.0 \cdot 0.18 \\&= 0.18\end{aligned}$$

Reason: By additionally considering the fix costs, the results are in the risk seeking part of the utility function of Mr. Berger.

1.3.3 Decision Matrix and Risk Utility

Question 23

a)

$$\begin{aligned}EV(a_1) &= 0.2 \cdot 70 + 0.5 \cdot 20 + 0.3 \cdot 5 \\ &= 25.5\end{aligned}$$

$$\begin{aligned}EV(a_2) &= 0.2 \cdot 40 + 0.5 \cdot 30 + 0.3 \cdot 25 \\ &= 30.5\end{aligned}$$

$$\begin{aligned}EV(a_3) &= 0.2 \cdot 10 + 0.5 \cdot 30 + 0.3 \cdot 50 \\ &= 32\end{aligned}$$

The production of the series Dolphin will be chosen. Reason: $EV(a_3) > EV(a_2) > EV(a_1)$. Risk neutrality is implied by corporate management.

b)

$$\begin{aligned}EU(a_1) &= 0.2 \cdot u(70) + 0.5 \cdot u(20) + 0.3 \cdot u(5) \\ &= 0.2 \cdot 1 + 0.5 \cdot 0.053 + 0.3 \cdot 0 \\ &= 0.2265\end{aligned}$$

$$\begin{aligned}EU(a_2) &= 0.2 \cdot u(40) + 0.5 \cdot u(30) + 0.3 \cdot u(25) \\ &= 0.2 \cdot 0.2899 + 0.5 \cdot 0.1479 + 0.3 \cdot 0.0946 \\ &= 0.1603\end{aligned}$$

$$\begin{aligned}EU(a_3) &= 0.2 \cdot u(10) + 0.5 \cdot u(30) + 0.3 \cdot u(50) \\ &= 0.2 \cdot 0.0059 + 0.5 \cdot 0.1479 + 0.3 \cdot 0.4792 \\ &= 0.2188\end{aligned}$$

The production of the series Tiger would be rational. Reason: $EU(a_1) > EU(a_3) > EU(a_2)$

Question 24

a) Expected value criterion

b) (1)

$$\begin{aligned}EU(a_1) &= p_1 \cdot u(6.3) + 0.2 \cdot u(4.5) + (0.8 - p_1) \cdot u(3.0) \\&= p_1 \cdot 0 + 0.2 \cdot 0.296 + (0.8 - p_1) \cdot 1 \\&= 0.8592 - p_1\end{aligned}$$

$$\begin{aligned}EU(a_3) &= p_1 \cdot u(3.5) + 0.2 \cdot u(4.7) + (0.8 - p_1) \cdot u(5.6) \\&= p_1 \cdot 0.72 + 0.2 \cdot 0.235 + (0.8 - p_1) \cdot 0.045 \\&= 0.083 + 0.675 \cdot p_1\end{aligned}$$

$$\begin{aligned}EU(a_1) &\stackrel{!}{=} EU(a_3) \\p_1 &= 0.46\end{aligned}$$

(2)

$$\begin{aligned}EU(a_2) &= p_1 \cdot u(5.1) + 0.2 \cdot u(4.8) + (0.8 - p_1) \cdot u(3.9) \\&= p_1 \cdot 0.132 + 0.2 \cdot 0.207 + (0.8 - p_1) \cdot 0.529 \\&= 0.4646 - 0.397 \cdot p_1\end{aligned}$$

$$\begin{aligned}EU(a_2) &\stackrel{!}{>} EU(a_1) \\p_1 &> 0.654\end{aligned}$$

$$\begin{aligned}EU(a_2) &\stackrel{!}{>} EU(a_3) \\p_1 &< 0.356\end{aligned}$$

The bond would never be chosen as a financial tool no matter probability p_1 .

1.3.4 Decision matrix and μ - σ Rule

Question 25

Criteria	Value
Level of uncertainty	risk
Number of objectives	1
Number of decision makers	1 decision maker with any risk attitude
Timeline	static
Behavior of opponent	random

Question 26

a)

$$\mu(a_i) = \sum_j p_j \cdot e_{i,j}$$

$$\sigma^2(a_i) = \left(\sum_j p_j \cdot e_{i,j}^2 \right) - \mu(a_i)^2$$

$$\sigma(a_i) = \sqrt{\sigma^2(a_i)}$$

	$\mu(a_i)$	$\sigma^2(a_i)$	$\sigma(a_i)$
Alternative A	-10	2900	53.85
Alternative B	6.4	32.64	5.71
Alternative C	4	0	0

b) Compare to solution of Question 27.

c)

	i) $\Phi(a_i)$	ii) $\Phi(a_i)$	iii) $\Phi(a_i)$
Alternative A	43.85	-10	-63.85
Alternative B	12.11	6.4	0.69
Alternative C	4	4	4

- i) M. S. acts risk seeking and chooses alternative A.
- ii) M. S. acts risk neutral and chooses alternative B.
- iii) M. S. acts risk averse and chooses alternative C.

Question 27

a)

$$\begin{aligned}\mu(a_1) &= 0.2 \cdot 80 + 0.5 \cdot 120 + 0.2 \cdot 130 + 0.1 \cdot 90 \\ &= 111\end{aligned}$$

$$\begin{aligned}\mu(a_2) &= 0.2 \cdot 110 + 0.5 \cdot 120 + 0.2 \cdot 100 + 0.1 \cdot 120 \\ &= 114\end{aligned}$$

$$\begin{aligned}\mu(a_3) &= 0.2 \cdot 90 + 0.5 \cdot 120 + 0.2 \cdot 115 + 0.1 \cdot 80 \\ &= 109\end{aligned}$$

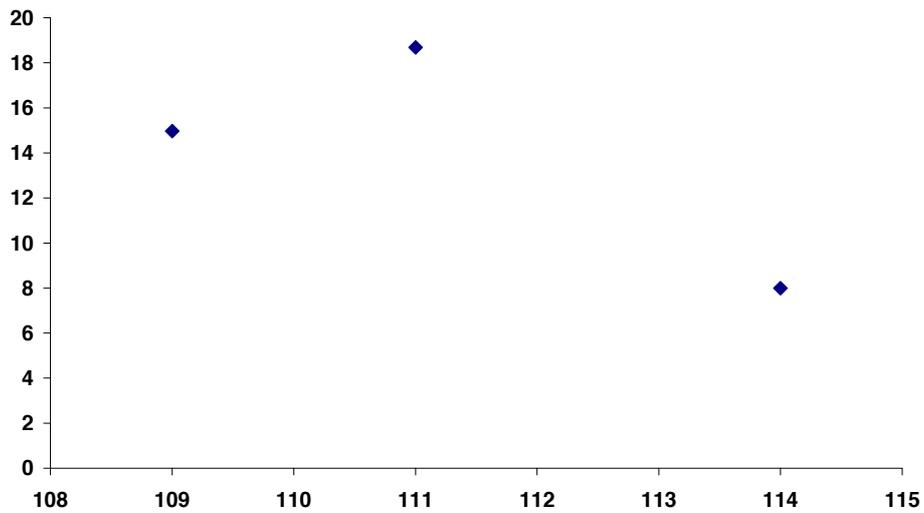
$$\begin{aligned}\sigma^2(a_1) &= 0.2 \cdot 80^2 + 0.5 \cdot 120^2 + 0.2 \cdot 130^2 + 0.1 \cdot 90^2 - 111^2 \\ &= 349\end{aligned}$$

$$\begin{aligned}\sigma^2(a_2) &= 0.2 \cdot 110^2 + 0.5 \cdot 120^2 + 0.2 \cdot 100^2 + 0.1 \cdot 120^2 - 114^2 \\ &= 64\end{aligned}$$

$$\begin{aligned}\sigma^2(a_3) &= 0.2 \cdot 90^2 + 0.5 \cdot 120^2 + 0.2 \cdot 115^2 + 0.1 \cdot 80^2 - 109^2 \\ &= 224\end{aligned}$$

	$\mu(a_i)$	$\sigma^2(a_i)$	$\sigma(a_i)$
Security A	111	349	18.68
Security B	114	64	8.00
Security C	109	224	14.97

b)



c)

(1) Compare to solution of Question 26 c.

(2)

$$\begin{aligned}\Phi(a_1) &= \frac{3}{2} \cdot \mu(a_1) + 2 \cdot \sigma(a_1) \\ &= \frac{3}{2} \cdot 111 + 2 \cdot 18.68 \\ &= 203.86\end{aligned}$$

$$\begin{aligned}\Phi(a_2) &= \frac{3}{2} \cdot \mu(a_2) + 2 \cdot \sigma(a_2) \\ &= \frac{3}{2} \cdot 114 + 2 \cdot 8.00 \\ &= 187\end{aligned}$$

$$\begin{aligned}\Phi(a_3) &= \frac{3}{2} \cdot \mu(a_3) + 2 \cdot \sigma(a_3) \\ &= \frac{3}{2} \cdot 109 + 2 \cdot 14.97 \\ &= 193.44\end{aligned}$$

Herr Bert prefers security A. Reason: $\Phi(a_1) > \Phi(a_3) > \Phi(a_2)$

Question 28

- Decision maker is risk neutral.
- Decision maker has quadratic utility function and parameters are calibrated accordingly.
- Results are normally distributed.

Question 29

a)

$$\begin{aligned}\mu(a_1) &= 0.2 \cdot 80 + 0.5 \cdot 120 + 0.1 \cdot 130 + 0.2 \cdot 90 \\ &= 107\end{aligned}$$

$$\begin{aligned}\mu(a_2) &= 0.2 \cdot 110 + 0.5 \cdot 120 + 0.1 \cdot 100 + 0.2 \cdot 80 \\ &= 108\end{aligned}$$

$$\begin{aligned}\mu(a_3) &= 0.2 \cdot 180 + 0.5 \cdot 270 + 0.1 \cdot 300 + 0.2 \cdot 140 \\ &= 229\end{aligned}$$

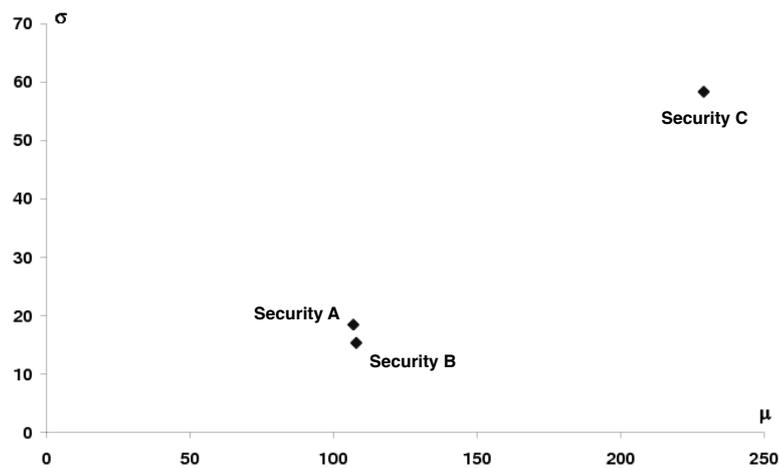
$$\begin{aligned}\sigma^2(a_1) &= 0.2 \cdot 80^2 + 0.5 \cdot 120^2 + 0.1 \cdot 130^2 + 0.2 \cdot 90^2 - 107^2 \\ &= 341\end{aligned}$$

$$\begin{aligned}\sigma^2(a_2) &= 0.2 \cdot 110^2 + 0.5 \cdot 120^2 + 0.1 \cdot 100^2 + 0.2 \cdot 80^2 - 108^2 \\ &= 236\end{aligned}$$

$$\sigma^2(a_3) = 0.2 \cdot 180^2 + 0.5 \cdot 270^2 + 0.1 \cdot 300^2 + 0.2 \cdot 140^2 - 229^2 = 3409$$

	$\mu(a_i)$	$\sigma^2(a_i)$	$\sigma(a_i)$
Security A	107	341	18.46
Security B	108	236	15.36
Security C	229	3409	55.21

b)



c)

(1) The decision maker is risk averse.

(2) a_4 : 3 times security A

a_5 : 3 times security B

- a_6 : 1 time security A and 1 time security C
- a_7 : 1 time security B and 1 time security C
- a_8 : 1 time security A and 2 times security B
- a_9 : 2 times security A and 1 time security B

	$\mu(a_i)$	$\sigma^2(a_i)$	$\Phi(a_i)$
a_4	321	3069	6358.62
a_5	324	2124	6437.52
a_6	336	5744	6605.12
a_7	337	4981	6640.38
a_8	323	1861	6422.78
a_9	322	2176	6396.48

The decision maker invests the 300 Euro into buying security B one time and security C one time. Reason: $\max_{i=4,\dots,9} \{\Phi(a_i)\} = \Phi(a_7)$

Note: For the random variables X, Y, Z , the constants a, b and for $Z = a \cdot X + b \cdot Y$ it holds: $\mu(Z) = a \cdot \mu(X) + b \cdot \mu(Y)$ and $\sigma^2(Z) = a^2 \cdot \sigma^2(X) + b^2 \cdot \sigma^2(Y) + 2 \cdot a \cdot b \cdot Cov(X, Y)$, where $Cov(X, Y)$ is the covariance of two random variables X and Y .

Example for a_9 :

2 times security A

1 time security B

$$\Rightarrow a = 2, b = 1$$

$$\mu(a_9) = 2 \cdot 107 + 1 \cdot 108 = 322$$

$$\sigma^2(a_9) = 2^2 \cdot 341 + 1^2 \cdot 236 + 2 \cdot 2 \cdot 1 \cdot Cov(A, B)$$

$$\begin{aligned} Cov(A, B) &= 0.2 \cdot (80 - 107) \cdot (110 - 108) \\ &+ 0.5 \cdot (120 - 107) \cdot (120 - 108) \\ &+ 0.1 \cdot (130 - 107) \cdot (100 - 108) \\ &+ 0.2 \cdot (90 - 107) \cdot (80 - 108) \end{aligned}$$

$$Cov(A, B) = 144$$

$$\Rightarrow \sigma^2(a_9) = 2^2 \cdot 341 + 1^2 \cdot 236 + 2 \cdot 2 \cdot 1 \cdot 144 = 2176$$

$$\Phi(a_9) = 20 \cdot 322 - 0.02 \cdot 2176 = 6396.48$$

1.4 Multi-Stage Decision Making

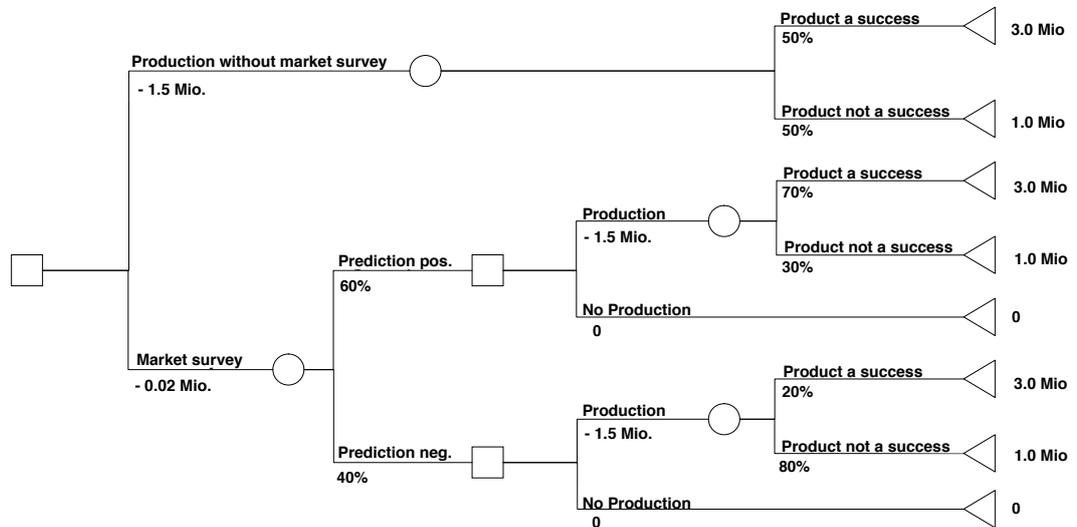
1.4.1 Decision Tree Analysis and Expected Value Criterion

Question 30

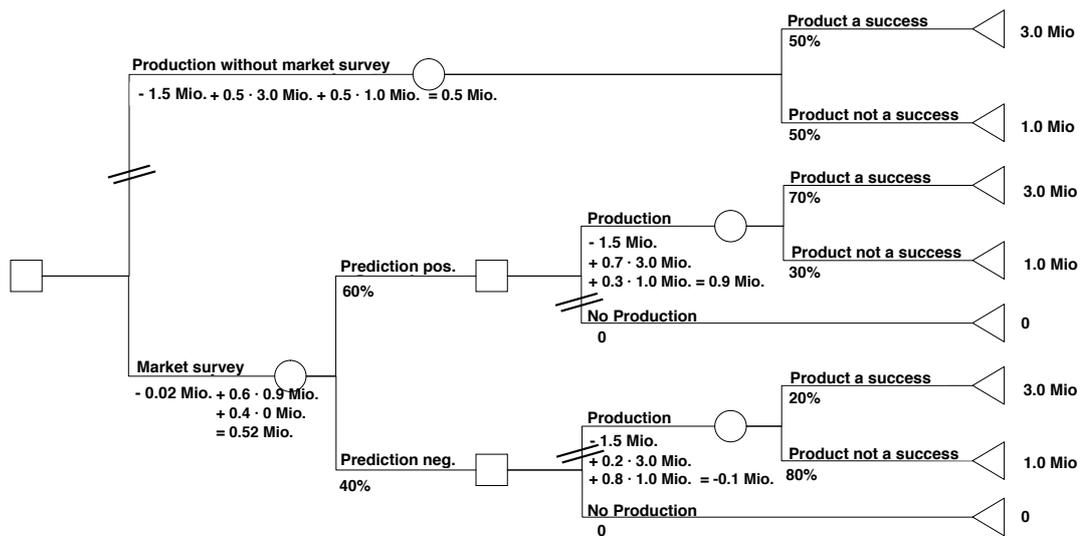
Criteria	Value
Level of uncertainty	risk
Number of objectives	1
Number of decision makers	1 decision maker with any risk attitude
Timeline	dynamic
Behavior of opponent	random

Question 31

a)



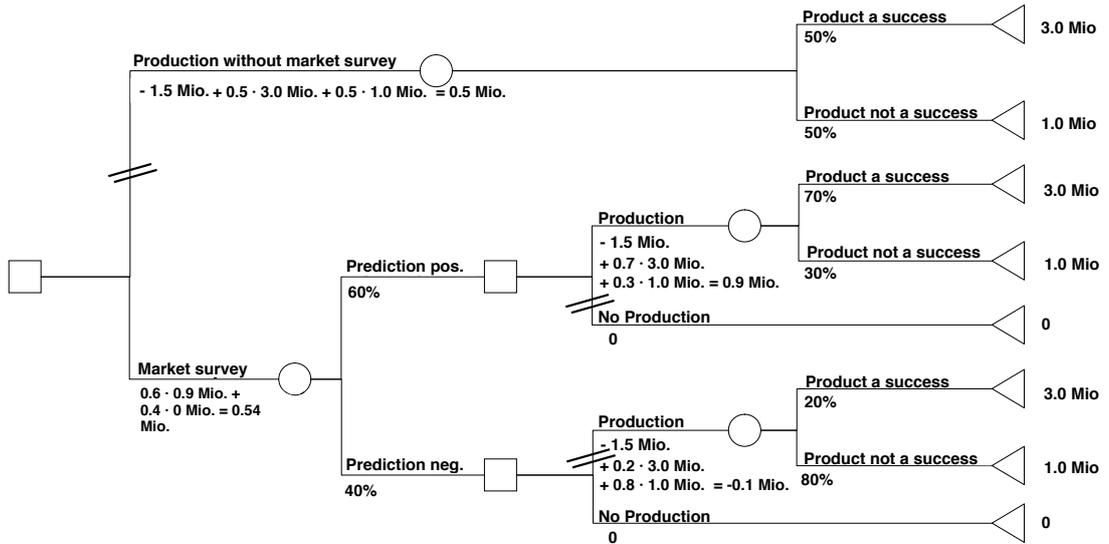
b)



The decision maker should do a market survey. If the survey's prediction is positive (negative), he should produce (not produce) the product.

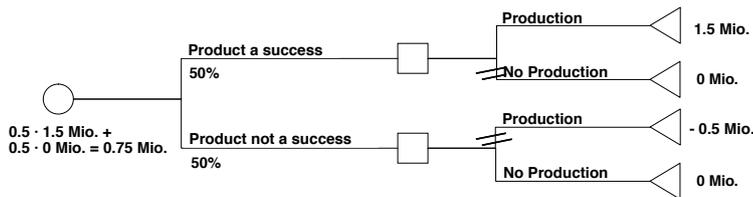
c)

The decision tree is solved another time under the assumption that the market analysis does not cost anything.



The entrepreneur should pay a maximum of 40,000 Euro for the information brought by the survey.

- d) The upper part of the decision tree (without market analysis) is solved again under the assumption that the investment decision is made after knowing which situation will occur.

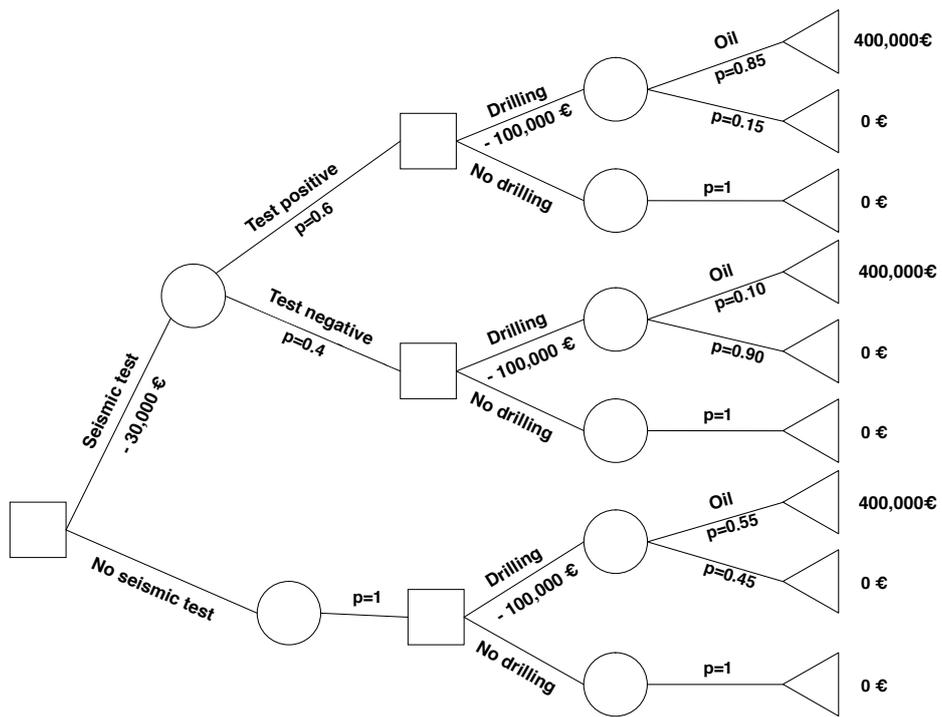


The entrepreneur should pay a maximum of 250,000 Euro for the perfect information.

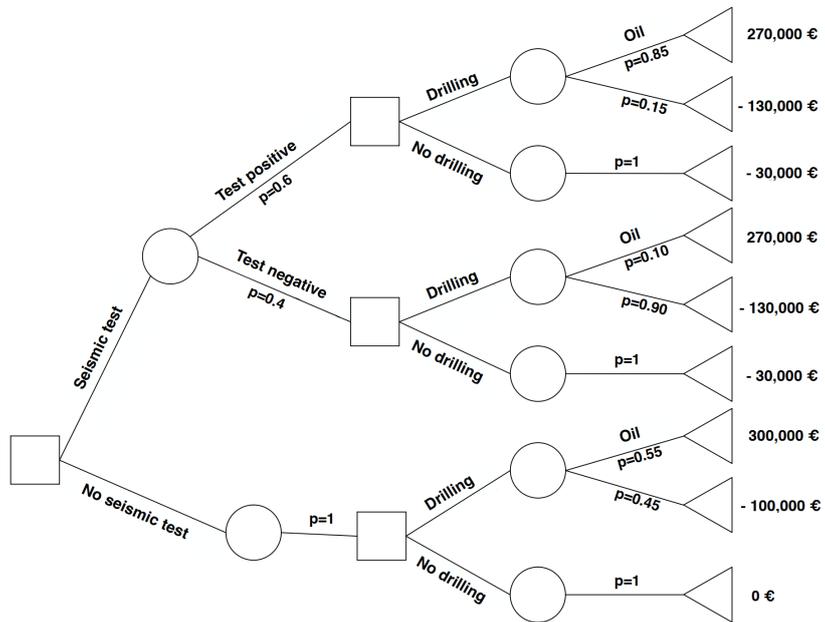
1.4.2 Decision Tree Analysis and Risk Utility

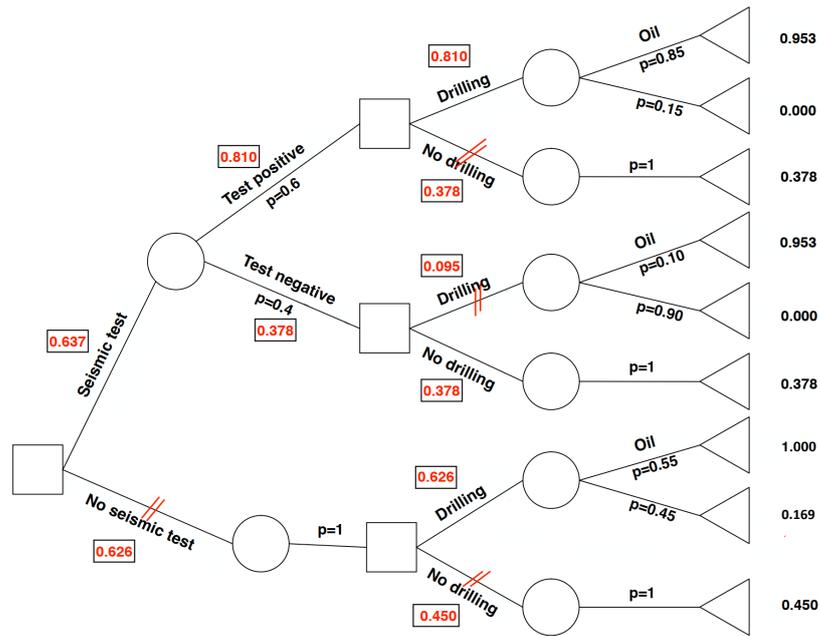
Question 32

- a)



b)





The owner should perform a seismic test. If the test is positive (negative), he should do (not do) a drilling.

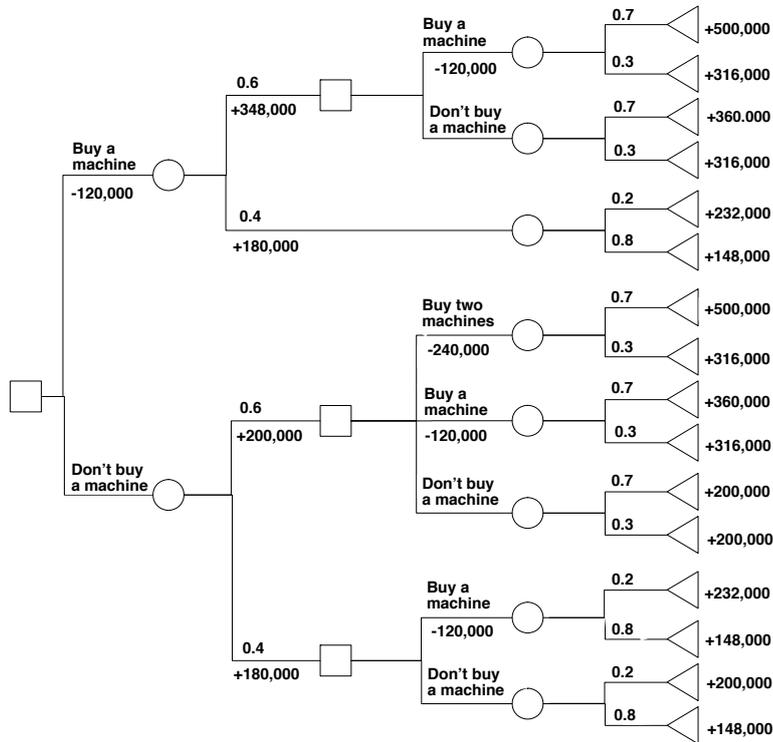
c)

$$0.626 \stackrel{!}{>} p \cdot 0.810 + (1 - p) \cdot 0.378$$

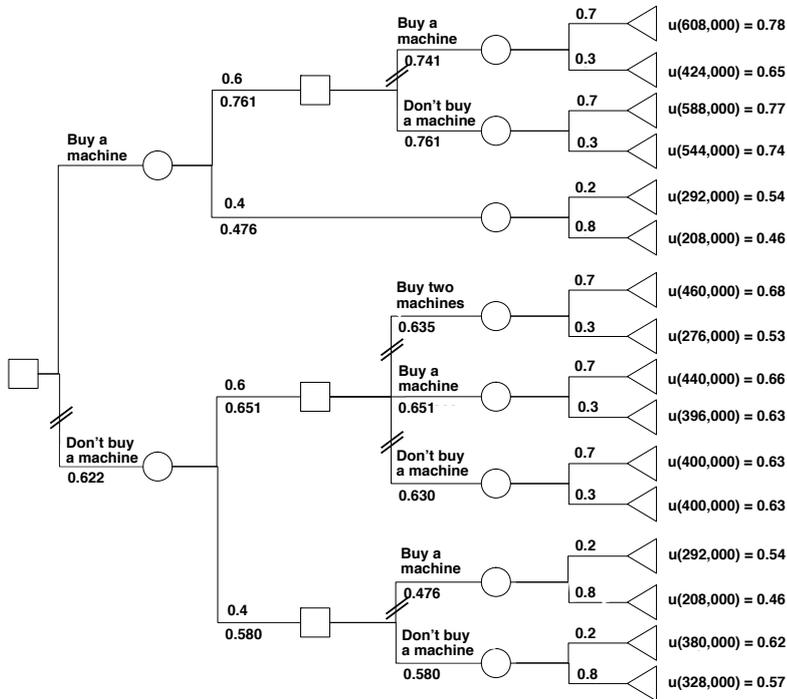
$$p < 57.4\%$$

Question 33

a)



b)



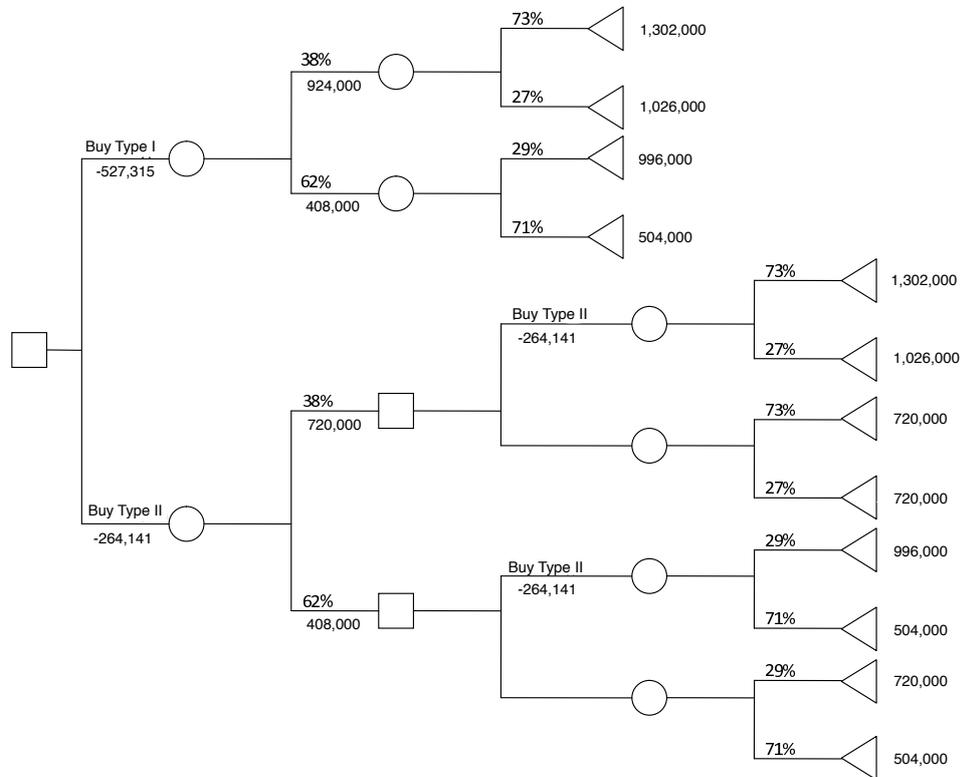
The entrepreneur should only buy a new machine at the beginning of the first year.

1.4.3 Decision Tree Analysis and Real Options

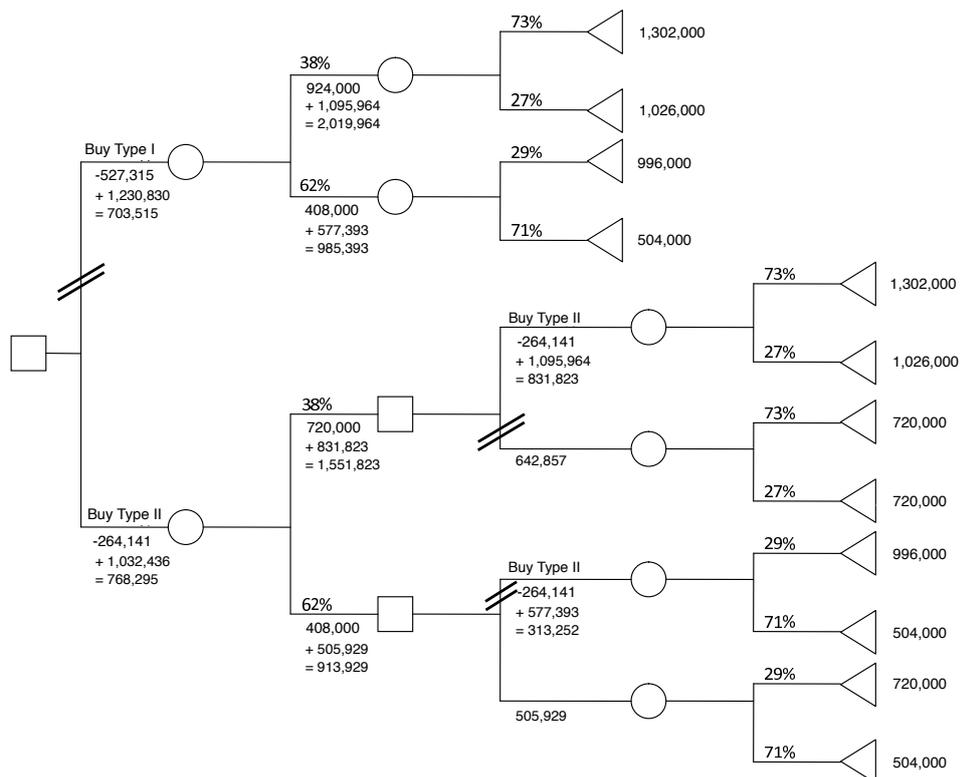
Question 34

Note: Here the net present value method is used to incorporate the discount interest rate of 12%.

a)



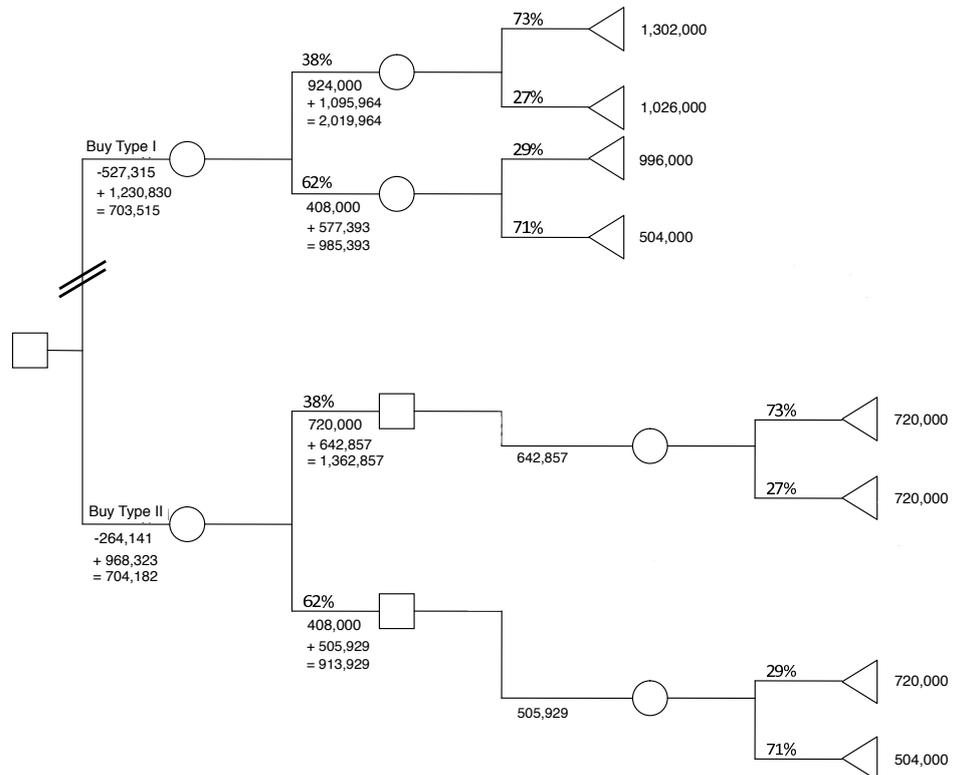
b)



The management should purchase a machine of type II at the beginning of the first year and only in case of high demand purchase another machine of type II at the beginning of the second year.

c)

The corresponding part of the decision tree is crossed out. The new maximum value is calculated and compared to the old value.



The real option has a value of $768,295 - 704,182 = 64,113$

1.5 Multi-Criteria Decision Making

1.5.1 Utility Analysis

Question 35

Criteria	Value
Level of uncertainty	deterministic
Number of objectives	> 1
Number of decision makers	1
Timeline	static
Behavior of opponent	random

Question 36

1. Determination of objectives
2. Determination and normalization of the weights of the objectives
3. Determination of the outcome values
4. Determination of the utility function for every objective
5. Determination of the expected utility of every alternative
6. Selection of alternative with highest expected utility

Question 37

a)

1. Determination of objectives (*refer to question formulation*)
2. Determination and normalization of the weights of the objectives

$$w_1 = \frac{6}{6 + 8 + 4 + 2} = 0.3$$

$$w_2 = \frac{8}{6 + 8 + 4 + 2} = 0.4$$

$$w_3 = \frac{4}{6 + 8 + 4 + 2} = 0.2$$

$$w_4 = \frac{2}{6 + 8 + 4 + 2} = 0.1$$

3. Determination of the outcome values (*see question formulation*)

4. Determination of the utility function for every objective

Costs

(a) $e_1^+ = 35,000$ and $e_1^- = 50,000$

(b) $u_1(35,000) = 1$ and $u_1(50,000) = 0$

(c) A linear utility function $u_1(e_{1j}) = a + b \cdot e_{1j}$ is needed

$$b = \frac{u_1(35,000) - u_1(50,000)}{35,000 - 50,000} = -\frac{1}{15,000}$$

$$a = 1 + \frac{1}{15,000} \cdot 35,000 = \frac{10}{3}$$

We get the linear utility function $u_1(e_{i1}) = \frac{10}{3} - \frac{1}{15,000} \cdot e_{i1}$

Risk

Transformation of qualitative values with low=3, medium=2 and high=1

(a) $e_2^+ = 3$ and $e_2^- = 1$

(b) $u_2(3) = 1$ and $u_2(1) = 0$

(c) $b = \frac{u_2(1) - u_2(3)}{1 - 3} = \frac{1}{2}$

$$a = 0 - \frac{1}{2} \cdot 1 = -\frac{1}{2}$$

We get the linear utility function $u_2(e_{i2}) = -\frac{1}{2} + \frac{1}{2} \cdot e_{i2}$

Increased Market Share

(a) $e_3^+ = 6\%$ and $e_3^- = 4\%$

(b) $u_3(6\%) = 1$ and $u_3(4\%) = 0$

(c) $b = \frac{u_3(4\%) - u_3(6\%)}{4\% - 6\%} = \frac{1}{2\%}$

$$a = 0 - \frac{1}{2\%} \cdot 4\% = -2$$

We get the linear utility function $u_3(e_{i3}) = -2 + \frac{1}{2\%} \cdot e_{i3}$

Prestige

Transformation of qualitative values with high=2 and medium=1

$$(a) e_4^+ = 2 \text{ and } e_4^- = 1$$

$$(b) u_4(2) = 1 \text{ and } u_4(1) = 0$$

$$(c) b = \frac{u_4(1) - u_4(2)}{1 - 2} = 1$$

$$a = 0 - 1 \cdot 1 = -1$$

We get the linear utility function $u_4(e_{i4}) = -1 + 1 \cdot e_{i4}$

5. Determination of the expected utility of every alternative

$$\begin{aligned} S(a_1) &= w_1 \cdot u_1(e_{11}) + w_2 \cdot u_2(e_{12}) + w_3 \cdot u_3(e_{13}) + w_4 \cdot u_4(e_{14}) \\ &= 0.3 \cdot u_1(50,000) + 0.4 \cdot u_2(3) + 0.2 \cdot u_3(5\%) + 0.1 \cdot u_4(1) \\ &= 0.3 \cdot 0 + 0.4 \cdot 1 + 0.2 \cdot \frac{1}{2} + 0.1 \cdot 0 \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} S(a_2) &= w_1 \cdot u_1(e_{21}) + w_2 \cdot u_2(e_{22}) + w_3 \cdot u_3(e_{23}) + w_4 \cdot u_4(e_{24}) \\ &= 0.3 \cdot u_1(35,000) + 0.4 \cdot u_2(1) + 0.2 \cdot u_3(4\%) + 0.1 \cdot u_4(1) \\ &= 0.3 \cdot 1 + 0.4 \cdot 0 + 0.2 \cdot 0 + 0.1 \cdot 0 \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} S(a_3) &= w_1 \cdot u_1(e_{31}) + w_2 \cdot u_2(e_{32}) + w_3 \cdot u_3(e_{33}) + w_4 \cdot u_4(e_{34}) \\ &= 0.3 \cdot u_1(48,000) + 0.4 \cdot u_2(2) + 0.2 \cdot u_3(6\%) + 0.1 \cdot u_4(2) \\ &= 0.3 \cdot \frac{2}{15} + 0.4 \cdot \frac{1}{2} + 0.2 \cdot 1 + 0.1 \cdot 1 \\ &= 0.54 \end{aligned}$$

6. Selection of alternative with highest expected utility, meaning select project C

b) Compare project A and project B

$$w_1 \cdot 0 + w_2 \cdot 1 + w_3 \cdot \frac{1}{2} + w_4 \cdot 0 \stackrel{!}{>} w_1 \cdot 1 + w_2 \cdot 0 + w_3 \cdot 0 + w_4 \cdot 0$$

$$\frac{8}{16 + g_3} \cdot 1 + \frac{g_3}{16 + g_3} \cdot \frac{1}{2} > \frac{6}{16 + g_3} \cdot 1$$

$$8 + \frac{1}{2} \cdot g_3 > 6$$

$$g_3 > -4$$

Compare project A and project C

$$\begin{aligned}
 w_1 \cdot 0 + w_2 \cdot 1 + w_3 \cdot \frac{1}{2} + w_4 \cdot 0 & \stackrel{!}{>} w_1 \cdot \frac{2}{15} + w_2 \cdot \frac{1}{2} + w_3 \cdot 1 + w_4 \cdot 1 \\
 \frac{8}{16 + g_3} \cdot 1 + \frac{g_3}{16 + g_3} \cdot \frac{1}{2} & > \frac{6}{16 + g_3} \cdot \frac{2}{15} + \frac{8}{16 + g_3} \cdot \frac{1}{2} + \frac{g_3}{16 + g_3} \cdot 1 + \frac{2}{16 + g_3} \cdot 1 \\
 8 + \frac{1}{2} \cdot g_3 & > \frac{4}{5} + 4 + g_3 + 2 \\
 g_3 & < 2.4
 \end{aligned}$$

Project A is selected if the weight of the objective „Increased market share“ lies in the interval $0 \leq g_3 < 2.4$.

Question 38

a)

1. Determination of objectives

Objectives are 1) an early market introduction and 2) a high market share. Low development costs are not included as a objective, because the question formulation says they are „irrelevant“.

2. Determination and normalization of the weights of the objectives

Select a random weight for objective 2): $g_2 = 1$

Set weight of objective 1) to three times that value: $g_1 = 3$

Normalization:

$$w_1 = \frac{3}{3 + 1} = 0.75$$

$$w_2 = \frac{1}{3 + 1} = 0.25$$

3. Determination of the outcome values (*refer to question formulation*)

4. Determination of the utility function for every objective

Market Introduction

Transformation of qualitative values with early=3, medium=2 and late=1

(a) $e_1^+ = 3$ and $e_1^- = 1$

(b) $u_1(3) = 1$ and $u_1(1) = 0$

(c) $b = \frac{u_1(3) - u_1(1)}{3 - 1} = \frac{1}{2}$

$$a = 0 - \frac{1}{2} \cdot 1 = -\frac{1}{2}$$

We get the linear utility function $u_1(e_{i1}) = -\frac{1}{2} + \frac{1}{2} \cdot e_{i1}$

Market Share

(a) $e_2^+ = 10\%$ and $e_2^- = 4\%$

(b) $u_2(10\%) = 1$ and $u_2(4\%) = 0$

(c) $b = \frac{u_2(10\%) - u_2(4\%)}{10\% - 4\%} = \frac{1}{6\%}$

$$a = 0 - \frac{1}{6\%} \cdot 4\% = -\frac{2}{3}$$

We get the linear utility function $u_2(e_{i2}) = -\frac{2}{3} + \frac{1}{6\%} \cdot e_{i2}$

5. Determination of the expected utility of every alternative

$$\begin{aligned} S(a_1) &= w_1 \cdot u_1(e_{11}) + w_2 \cdot u_2(e_{12}) \\ &= 0.75 \cdot u_1(3) + 0.25 \cdot u_2(4\%) \\ &= 0.75 \cdot 1 + 0.25 \cdot 0 \\ &= 0.75 \end{aligned}$$

$$\begin{aligned} S(a_2) &= w_1 \cdot u_1(e_{21}) + w_2 \cdot u_2(e_{22}) \\ &= 0.75 \cdot u_1(2) + 0.25 \cdot u_2(10\%) \\ &= 0.75 \cdot 0.5 + 0.25 \cdot 1 \\ &= 0.625 \end{aligned}$$

$$\begin{aligned} S(a_3) &= w_1 \cdot u_1(e_{31}) + w_2 \cdot u_2(e_{32}) \\ &= 0.75 \cdot u_1(1) + 0.25 \cdot u_2(8\%) \\ &= 0.75 \cdot 0 + 0.25 \cdot \frac{2}{3} \\ &= \frac{1}{6} \end{aligned}$$

6. Selection of alternative with highest expected utility, meaning select product A

b) Representation of the utility value $S(a_i)$ of the alternatives a_i as a function of the weight w_2 of objective 2):

$$\begin{aligned} S(a_1) &= (1 - w_2) \cdot 1 + w_2 \cdot 0 \\ &= 1 - w_2 \end{aligned}$$

$$\begin{aligned} S(a_2) &= (1 - w_2) \cdot 0.5 + w_2 \cdot 1 \\ &= 0.5 + 0.5 \cdot w_2 \end{aligned}$$

Determination of the intersection:

$$S(a_1) = 1 - w_2 \stackrel{!}{=} 0.5 + 0.5 \cdot w_2 = S(a_2)$$

$$w_2 = \frac{1}{3}$$

- c) If the expected market share of product B is only 9%, B would still be assigned the highest utility and A would be assigned the lowest utility for the market share when determining the utility function. The expected utility will therefore stay the same for both products and the company will be indifferent between A and B.

A risk seeking company would still assign a value of 1 and 0 to the highest and lowest market share. An earlier market introduction will still be assigned a utility of 1. As risk seeking decision makers possess a convex utility function, it holds:

$$u_{riskseeking}(x) \leq u_{riskneutral}(x) \text{ for all } x \in [e^-, e^+]$$

A risk seeking company will therefore assign a smaller or equal utility to the value „medium“ of the objective market introduction compared to a risk neutral company. It would select product A or stay indifferent.

Question 39

a)

1. Determination of objectives (*refer to question formulation*)
2. Determination and normalization of the weights of the objectives

$$w_1 = \frac{5}{5 + 3 + 2} = 0.5$$

$$w_2 = \frac{3}{5 + 3 + 2} = 0.3$$

$$w_3 = \frac{2}{5 + 3 + 2} = 0.2$$

3. Determination of the outcome values (*refer to question formulation*)
4. Determination of the utility function for every objective

Costs

(a) $e_1^+ = 0$ and $e_1^- = 100$

(b) $u_1(0) = 1$ and $u_1(100) = 0$

(c) A linear utility function $u_1(e_{1j}) = a + b \cdot e_{1j}$ is needed

$$b = \frac{u_1(0) - u_1(100)}{0 - 100} = -\frac{1}{100}$$

$$a = 1 + \frac{1}{100} \cdot 0 = 1$$

We get the linear utility function $u_1(e_{i1}) = 1 - \frac{1}{100} \cdot e_{i1}$

Usability

Transformation of qualitative values:

Qualitative values	Quantitative values
very difficult	1
difficult	2
medium	3
easy	4
very easy	5

(a) $e_2^+ = 5$ and $e_2^- = 1$

(b) $u_2(5) = 1$ and $u_2(1) = 0$

(c) $b = \frac{u_2(1) - u_2(5)}{1 - 5} = \frac{1}{4}$

$$a = 0 - \frac{1}{4} \cdot 1 = -\frac{1}{4}$$

We get the linear utility function $u_2(e_{i2}) = -\frac{1}{4} + \frac{1}{4} \cdot e_{i2}$

Functions

Transformation of qualitative values:

Qualitative values	Quantitative values
very little	1
little	2
medium	3
many	4

(a) $e_3^+ = 4$ and $e_3^- = 1$

(b) $u_3(4) = 1$ and $u_3(1) = 0$

(c) $b = \frac{u_3(1) - u_3(4)}{1 - 4} = \frac{1}{3}$

$$a = 0 - \frac{1}{3} \cdot 1 = -\frac{1}{3}$$

We get the linear utility function $u_3(e_{i3}) = -\frac{1}{3} + \frac{1}{3} \cdot e_{i3}$

5. Determination of the expected utility of every alternative

Alternative	Costs	Usability	Functions	$S(a_i)$
Super Type	0	0	1	0.200
Easywrite	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{2}{3}$	0,258
Right Word	$\frac{1}{2}$	1	$\frac{2}{3}$	0.683
Lexico	1	$\frac{1}{2}$	$\frac{1}{3}$	0.716
Ultraword	$\frac{8}{10}$	1	0	0.700
Keywrite	$\frac{6}{10}$	$\frac{3}{4}$	$\frac{2}{3}$	0.658
Fastwrite	$\frac{3}{20}$	$\frac{1}{2}$	1	0.425

6. Selection of alternative with highest expected utility, meaning select software „Lexico“

- b) Representation of utility value $S(a_i)$ of the alternatives a_i as a function depending on weight w_1 of objective “costs“:

$$\begin{aligned} S(a_1) &= w_1 \cdot 0 + (1 - w_1) \cdot 1 \\ &= 1 - w_1 \end{aligned}$$

$$\begin{aligned} S(a_2) &= w_1 \cdot 0,10 + (1 - w_1) \cdot \frac{2}{3} \\ &= \frac{2}{3} - 0,566 \cdot w_1 \end{aligned}$$

$$\begin{aligned} S(a_3) &= w_1 \cdot 0,50 + (1 - w_1) \cdot \frac{2}{3} \\ &= \frac{2}{3} - 0,166 \cdot w_1 \end{aligned}$$

$$\begin{aligned} S(a_4) &= w_1 \cdot 1 + (1 - w_1) \cdot \frac{1}{3} \\ &= \frac{1}{3} + \frac{2}{3} \cdot w_1 \end{aligned}$$

$$\begin{aligned} S(a_5) &= w_1 \cdot 0,80 + (1 - w_1) \cdot 0 \\ &= 0,8 \cdot w_1 \end{aligned}$$

$$\begin{aligned} S(a_6) &= w_1 \cdot 0,60 + (1 - w_1) \cdot \frac{2}{3} \\ &= \frac{2}{3} - 0,066 \cdot w_1 \end{aligned}$$

$$\begin{aligned} S(a_7) &= w_1 \cdot 0,15 + (1 - w_1) \cdot 1 \\ &= 1 - 0,85 \cdot w_1 \end{aligned}$$

Determination of intercept points and graphical representation:

$$S(a_7) = 1 - 0.85 \cdot w_1 \stackrel{!}{=} \frac{2}{3} - 0.066 \cdot w_1 = S(a_6)$$

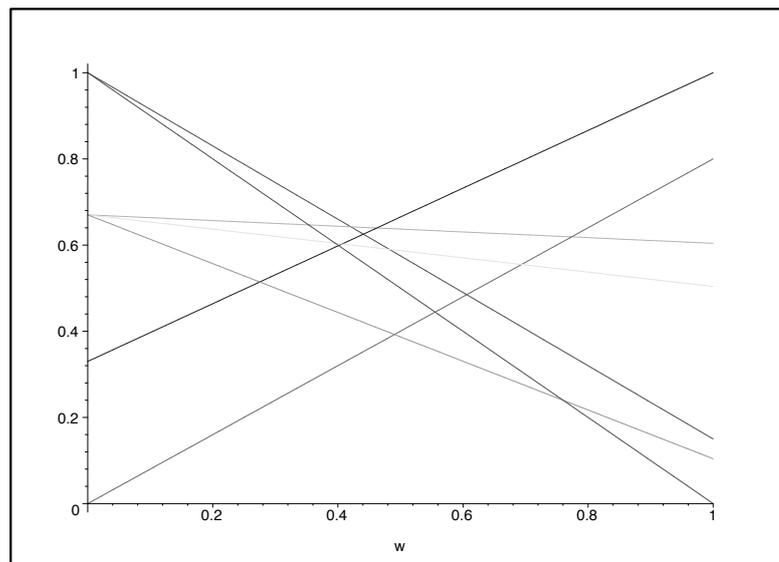
$$w_1 = 0.425$$

$$S(a_6) = \frac{2}{3} - 0.066 \cdot w_1 \stackrel{!}{=} \frac{1}{3} + \frac{2}{3} \cdot w_1 = S(a_4)$$

$$w_1 = 0.455$$

Recommendation:

- If the weight of the objective “costs“ is in the interval $0 \leq w_1 \leq 0.425$, the product „Fastwrite“ is recommended.
- If the weight of the objective “costs“ is in the interval $0.425 < w_1 \leq 0.455$, the product „Keywrite“ is recommended.
- If the weight of the objective “costs“ is in the interval $0.455 < w_1 \leq 1$, the product „Lexico“ is recommended.



1.6 Monte Carlo Simulation

Question 40

1. Creation of a deterministic model
2. Transformation of the deterministic model to a stochastic model
 - (a) Determination of the stochastic input data.
 - (b) Determination of the density function for the stochastic input data.
 - (c) Determination of the output cells for which the density function is supposed to be logged.
3. Execute simulation
 - (a) Model is played n times.
 - (b) For every simulation every stochastic input is a random number determined on the basis of the specified distribution.
 - (c) The output quantity is calculated and logged through the model every time.
4. Analyzing the result
 - (a) The result of the simulation is the density function of the output quantity.
 - (b) Analysis through determination of the density function and the static key figures.

Question 41

- a)
$$G(x) = \begin{cases} 1.50 \cdot 13 - 1.20 \cdot 13 & \text{if } x \geq 13 \\ 1.50 \cdot x - 1.20 \cdot 13 & \text{if } x < 13 \end{cases}$$
- b) Deterministic:
- purchase price
 - selling price
 - Number of daily bought copies
- Stochastic:
- Number of daily demanded copies (Distribution in given table)
- c) *No solution available!*

Question 42

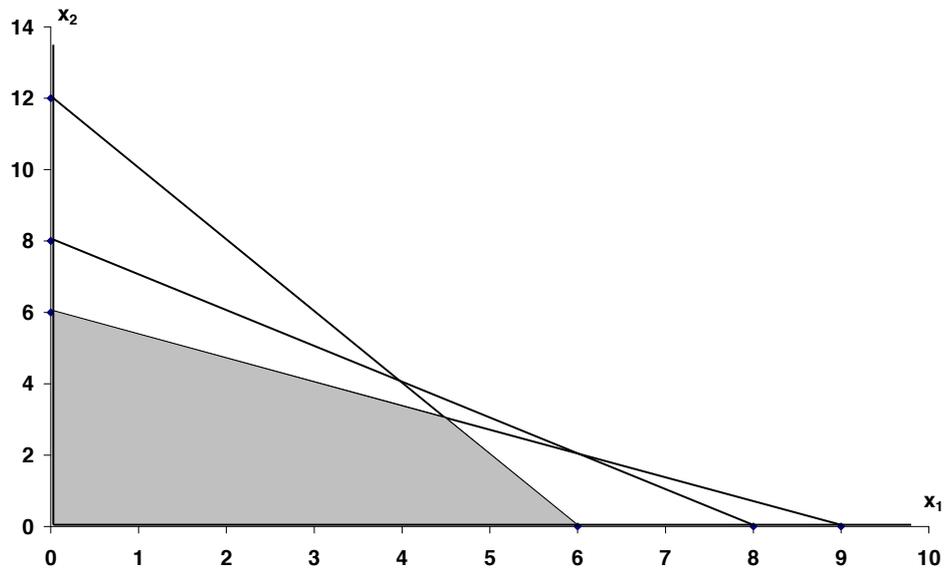
- a)
$$G(x,n) = \begin{cases} \min\{19,21-n\} \cdot 150 - \max\{0,21-n-19\} \cdot 175 & \text{if } x \geq 21 \\ \min\{19,x-n\} \cdot 150 - \max\{0,x-n-19\} \cdot 175 & \text{if } x < 21 \end{cases}$$
- b) Deterministic:
- Revenue generated with every passenger transported
 - Costs generated with every passenger not transported
 - Capacity of the airplane
 - Number of booked passengers
- Stochastic:
- Demand (Distribution in given table)
 - Number of no-shows (Poisson-Distribution with $\mu = 2$)
- c) *No solution available!*

2 Linear Programming

2.1 Modeling and Graphical Solutions

Question 43

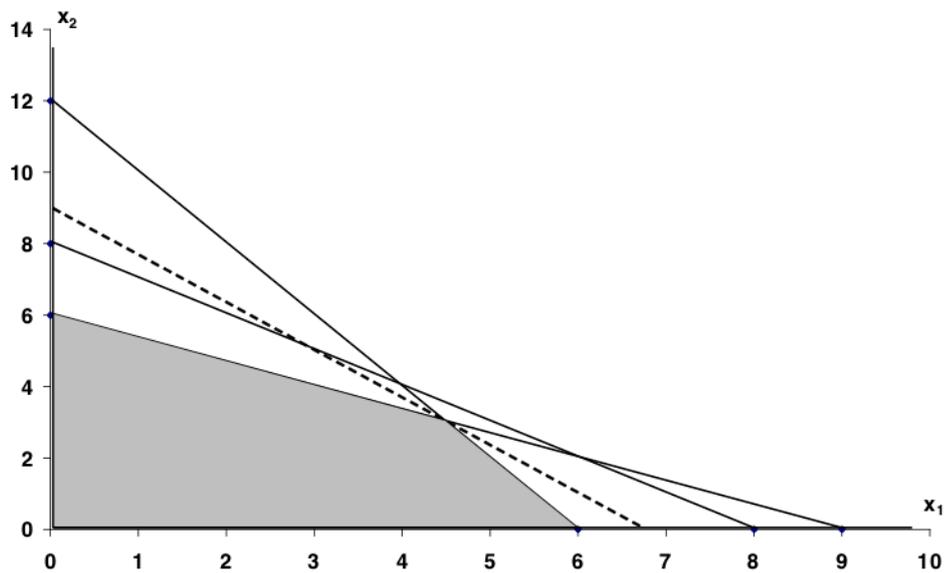
a)



b)

Point	Objective function value
(0,0)	0
(6,0)	240
(0,6)	180
$(\frac{9}{2}, 3)$	270

c)



The extreme point $(\frac{9}{2}, 3)$ of the feasible region with the objective function value of 270 is optimal (refer to figure above).

Question 44

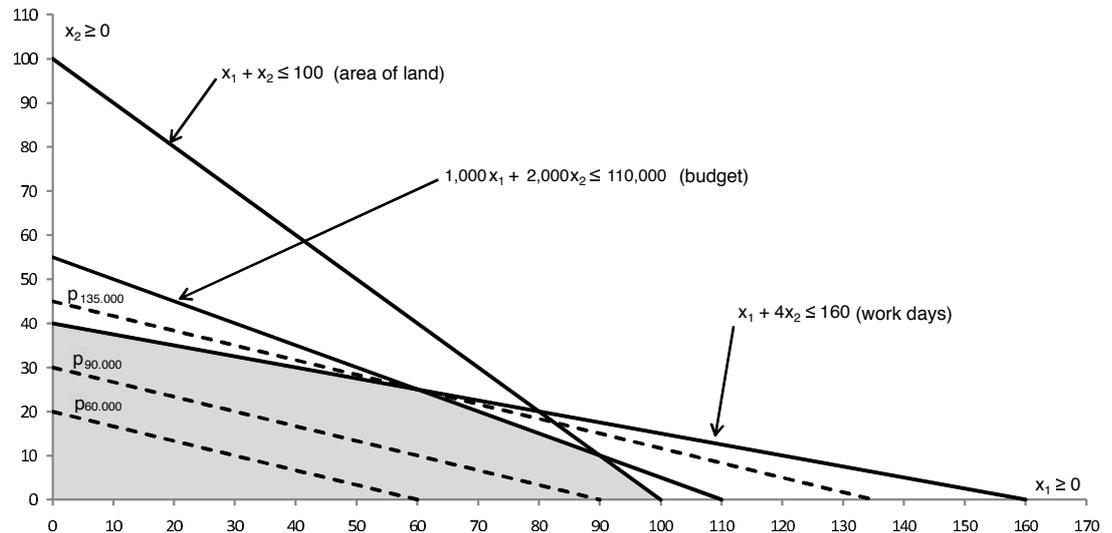
a) Maximize

$$z = 1,000 \cdot x_1 + 3,000 \cdot x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 100 \\ x_1 + 4 \cdot x_2 &\leq 160 \\ 1,000 \cdot x_1 + 2,000 \cdot x_2 &\leq 110,000 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- b) 1. Specification of the objective
- Maximize the total profit of the farmer.
2. Specification of the decision variables
- x_1 : area of land in ha, which is used to grow potatoes
 - x_2 : area of land in ha, which is used to grow cereal crops
3. Short description of every constraint
- A maximum of 100 ha of farming land is available to the farmer.
 - There are only 160 work days available.
 - The budget of 110,000 Euro cannot be exceeded.
 - The decision variables cannot be negative.
- c) refer to d)
- d)



The maximum profit of 135,000 Euro is reached through using 60ha for planting potatoes and 25ha for planting cereal crops.

Question 45

a) Maximize

$$z = 3 \cdot x_1 + 5 \cdot x_2$$

subject to

$$x_1 + x_2 \leq 10$$

$$x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

b) 1. Specification of the objective

- Maximize the profit per day of the fish-processing company.

2. Specification of the decision variables

- x_1 : pure crab meat sold in kg per day
- x_2 : crab meat used to produce crab salad in kg per day

3. Formulation of the objective function, which has to be a linear function containing the decision variables and is maximized / minimized.

- Maximize $P(x_1, x_2) = 3 \cdot x_1 + 5 \cdot x_2$

4. Short description of every constraint

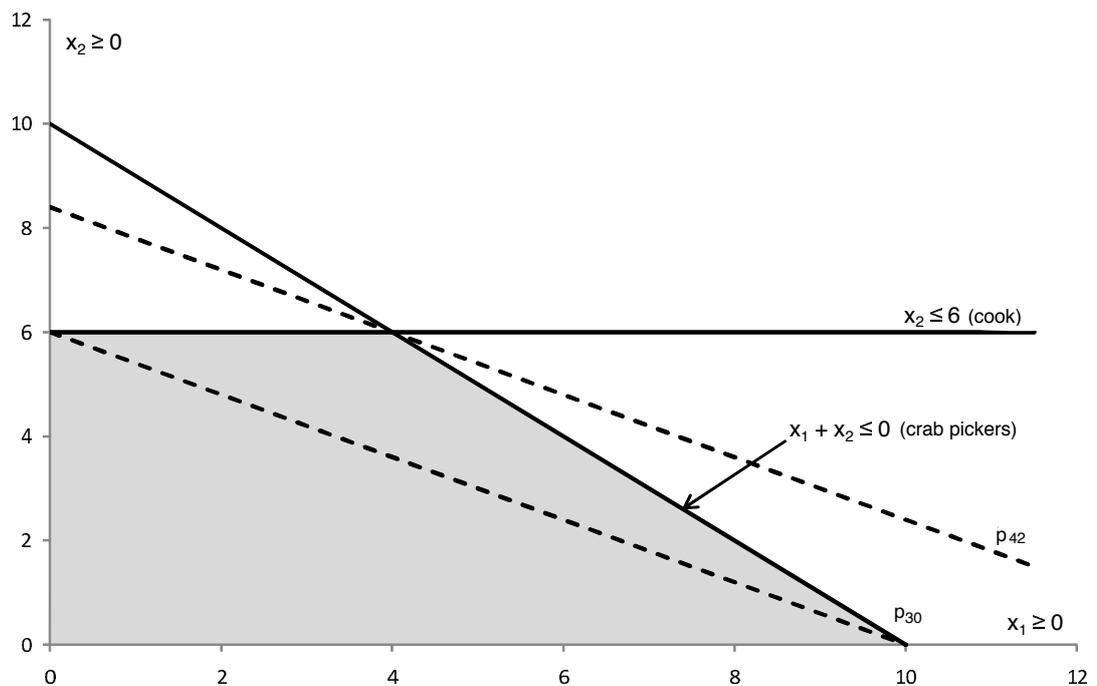
- The crab pickers are able to produce a maximum of 10kg of pure crab meat per day.
- The cook is able to produce a maximum of 6kg of crab salad per day.
- The decision variables cannot be negative.

5. Formulation of all constraints for the decision variables as linear equalities or inequalities.

- $x_1 + x_2 \leq 10$
- $x_2 \leq 6$
- $x_1, x_2 \geq 0$

c) refer to d)

d)



The maximum profit of 42 Euro per day is reached through producing and selling 6kg of crab salad and 4kg of pure crab meat per day.

Question 46

a) Minimize

$$z = 200 \cdot x_1 + 160 \cdot x_2$$

subject to

$$\begin{aligned} 6 \cdot x_1 + 2 \cdot x_2 &\geq 12 \\ 2 \cdot x_1 + 2 \cdot x_2 &\geq 8 \\ 4 \cdot x_1 + 12 \cdot x_2 &\geq 24 \\ x_1, x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

b) 1. Specification of the objective

- Minimize the operating costs of the mining company.

2. Specification of the decision variables

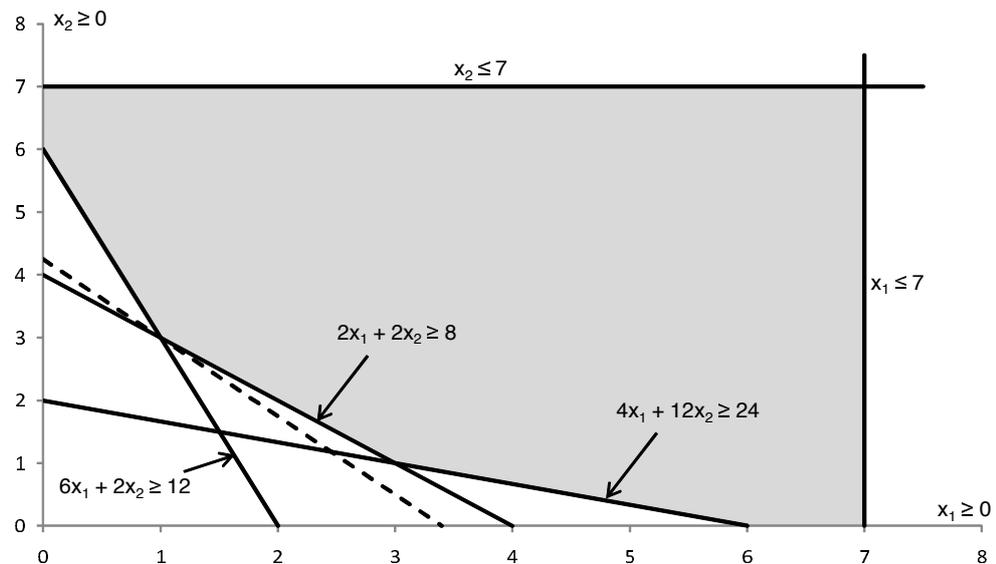
- x_1 : days per week mine 1 is operated
- x_2 : days per week mine 2 is operated

3. Short description of every constraint

- The demand of the smelter for coarse-grained ore has to be satisfied.
- The demand of the smelter for medium-grained ore has to be satisfied.
- The demand of the smelter for fine-grained ore has to be satisfied.
- Every mine can be operated a maximum of 7 days per week.
- The decision variables cannot be negative.

c) refer to d)

d)



Question 47

a) Maximize

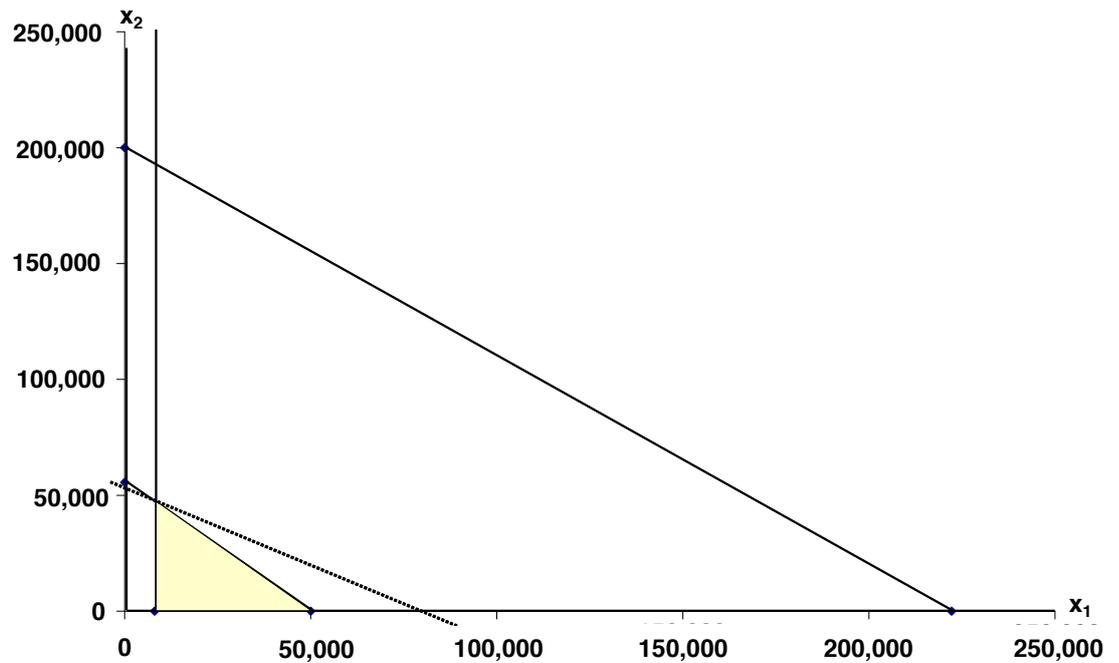
$$z = 1 \cdot x_1 + 1.5 \cdot x_2$$

subject to

$$\begin{aligned}0.2 \cdot x_1 + 0.18 \cdot x_2 &\leq 10,000 \\0.009 \cdot x_1 + 0.01 \cdot x_2 &\leq 2,000 \\1 \cdot x_1 &\geq 8,000 \\x_1, x_2 &\geq 0\end{aligned}$$

- b)
1. Specification of the objective
 - Maximize the total revenue.
 2. Specification of the decision variables
 - x_1 : amount of produced liters of *Classic*
 - x_2 : amount of produced liters of *Free*
 3. Short description of every constraint
 - The available amount of malt cannot be exceeded.
 - The available amount of hop cannot be exceeded.
 - The advanced orders for *Classic* must be satisfied.
 - The decision variables cannot be negative.

d)



The maximum revenue of 78,000 Euro is reached by producing 8,000 liters of *Classic* and 46,666.67 liters of *Free*.

- e) The presented production plan can never be optimal as it does not represent an extreme point of the feasible region.

2.2 Simplex Algorithm: Basics

Question 48

The linear program has to be in canonical form, which means:

- The co-domain of the decision variables is ≥ 0 .
- All constraints excluding the non-negativity constraint are formulated as equalities.
- The right-hand side of every equation is non-negative.
- In every constraint there exists exactly one variable with coefficient +1, which possesses the coefficient 0 in all other constraints and the objective function.

Question 49

- a) $x_1 = 0$ and $x_2 = 0$ (non-basic variables) as well as $x_3 = 10$, $x_4 = 8$ and $x_5 = 4$ (basic variables) with objective function coefficient $z = 0$.
- b) The basic feasible solution is feasible, because when inserting the values of the decision variables into the constraints all of them are satisfied:

$$1 \cdot 0 + 2 \cdot 0 + 1 \cdot 10 + 0 \cdot 8 + 0 \cdot 4 = 10$$

$$2 \cdot 0 + \frac{3}{2} \cdot 0 + 0 \cdot 10 + 1 \cdot 8 + 0 \cdot 4 = 8$$

$$0 \cdot 0 + 1 \cdot 0 + 0 \cdot 10 + 0 \cdot 8 + 1 \cdot 4 = 4$$

It is not optimal, because the solution $x_1 = 1$, $x_2 = 0$, $x_3 = 9$, $x_4 = 6$ and $x_5 = 4$ is feasible as well and leads to a better objective function value of $z = 3$.

- c) 1. Increasing x_1 (x_2) by one unit increases the objective function value by three (two) units.
2. Variable x_1 is chosen as increasing x_1 leads to a greater increase of the objective function value.

3. Determining the upper bound of the value of x_1 by using the constraints of the linear program:

$$1 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = 10 \Rightarrow x_1 \leq 10$$

$$2 \cdot x_1 + \frac{3}{2} \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 = 8 \Rightarrow x_1 \leq 4$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 = 4 \Rightarrow x_1 \text{ is unbounded.}$$

4. Increasing x_1 up to the maximum value, here $x_1 = 4$.

5. Changing the values of the old basic variables to make the new basic solution feasible.

$$1 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = 10 \Rightarrow x_3 = 6$$

$$2 \cdot x_1 + \frac{3}{2} \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 = 8 \Rightarrow x_4 = 0$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 = 4 \Rightarrow x_5 = 4$$

Observation:

The non-basic variables x_1 became a basic variable, while the basic variable x_4 became a non-basic variable. This transition from a feasible basic solution to another feasible basic solution is called change of the basis.

6. Transformation of the LP into canonical form by using the Gauß-Jordan algorithm.

- The new basic variable (x_1) has to possess the coefficient 1 in the binding equation. Therefore you begin by multiplying the second equation with factor $\frac{1}{2}$. The result is:

$$1 \cdot x_1 + \frac{3}{4} \cdot x_2 + 0 \cdot x_3 + \frac{1}{2} \cdot x_4 + 0 \cdot x_5 = 4$$

- The new basic variable must have the coefficient 0 in all other equations. This is clearly not the case. Hence the (new) second equation is subtracted from the first equation. The result is:

$$0 \cdot x_1 + \frac{5}{4} \cdot x_2 + 1 \cdot x_3 - \frac{1}{2} \cdot x_4 + 0 \cdot x_5 = 6$$

- The new basic variable must have the coefficient 0 in the objective function. That is also not the case. The (new) second equation is therefore multiplied with the factor 3 and subtracted from the objective function. The result is:

$$0 \cdot x_1 - \frac{1}{4} \cdot x_2 + 0 \cdot x_3 - \frac{3}{2} \cdot x_4 + 0 \cdot x_5 = z - 12$$

The following is the linear program in canonical form related to the new basic feasible solution:

Maximize

$$0 \cdot x_1 - \frac{1}{4} \cdot x_2 + 0 \cdot x_3 - \frac{3}{2} \cdot x_4 + 0 \cdot x_5 = z - 12$$

subject to

$$0 \cdot x_1 + \frac{5}{4} \cdot x_2 + 1 \cdot x_3 - \frac{1}{2} \cdot x_4 + 0 \cdot x_5 = 6$$

$$1 \cdot x_1 + \frac{3}{4} \cdot x_2 + 0 \cdot x_3 + \frac{1}{2} \cdot x_4 + 0 \cdot x_5 = 4$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 = 4$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Question 50

- a) $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, (non-basic variables) as well as $x_4 = 5$ and $x_5 = 7$ (basic variables) with objective function value $z = 0$.
- b) The basic feasible solution is feasible, because when inserting the values of the decision variables into the constraints all of them are satisfied:

$$2 \cdot 0 + 2 \cdot 0 + 0 \cdot 0 + 1 \cdot 5 + 0 \cdot 7 = 5$$

$$1 \cdot 0 + 4 \cdot 0 + 4 \cdot 0 + 0 \cdot 5 + 1 \cdot 7 = 7$$

It is not optimal, because the solution $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 3$ and $x_5 = 6$ is feasible as well and leads to a better objective function value of $z = 6$.

c) (same procedure as Question 7)

Maximize

$$z = 0 \cdot x_1 - 18 \cdot x_2 - 4 \cdot x_3 - 3 \cdot x_4 + 0 \cdot x_5 + 15$$

subject to

$$\begin{aligned} 1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + \frac{1}{2} \cdot x_4 + 0 \cdot x_5 &= \frac{5}{2} \\ 0 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 - \frac{1}{2} \cdot x_4 + 1 \cdot x_5 &= \frac{9}{2} \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Question 51

a)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	25	6	3	5	1	
x_5	20	3	4	5		1
$-z$	0	3	1	4		

b) The basic solution is feasible as all values of the basic variables are greater or equal to zero, but not optimal as at least one non-basic variable possesses a objective function coefficient of greater zero.

c)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	25	6	3	5	1	
x_5	20	3	4	5		1
$-z$	0	3	1	4		

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	5	3	-1		1	-1
x_3	4	$\frac{3}{5}$	$\frac{4}{5}$	1		$\frac{1}{5}$
$-z$	-16	$\frac{3}{5}$	$-\frac{11}{5}$			$-\frac{4}{5}$

BV	Value	x_1	x_2	x_3	x_4	x_5
x_1	$\frac{5}{3}$	1	$-\frac{1}{3}$		$\frac{1}{3}$	$-\frac{1}{3}$
x_3	3		1	1	$-\frac{1}{5}$	$\frac{2}{5}$
$-z$	-17		-2		$-\frac{1}{5}$	$-\frac{3}{5}$

Question 52

a)

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	10	1	-1		1			
x_5	12	1		-1		1		
x_6	8	-1	1	1			1	
x_7	2	2	-1	1				1
$-z$	0	1	1	3				

b)

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	10	1	-1		1			
x_5	12	1		-1		1		
x_6	8	-1	1	1			1	
x_7	2	2	-1	1				1
$-z$	0	1	1	3				

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	10	1	-1		1			
x_5	14	3	-1			1		1
x_6	6	-3	2				1	-1
x_3	2	2	-1	1				1
$-z$	-6	-5	4					-3

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	13	$-\frac{1}{2}$			1		$\frac{1}{2}$	$-\frac{1}{2}$
x_5	17	$\frac{3}{2}$				1	$\frac{1}{2}$	$\frac{1}{2}$
x_2	3	$-\frac{3}{2}$	1				$\frac{1}{2}$	$-\frac{1}{2}$
x_3	5	$\frac{1}{2}$		1			$\frac{1}{2}$	$\frac{1}{2}$
$-z$	-18	1					-2	-1

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	18			1	1		1	
x_5	2			-3		1	-1	-1
x_2	18		1	3			2	1
x_1	10	1		2			1	1
$-z$	-28			-2			-3	-2

Question 53

a) Selection of x_2 as pivot column: Select column of NBV with highest positive objective function coefficient.

Selection of s_2 as pivot row: Select row with minimal value $\{\frac{\bar{b}_i}{\bar{a}_{is}} | \bar{a}_{is} > 0\}$.

Consequently variable s_2 will leave the basis.

b)

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	7	1	3	1		
s_2	2	1	2		1	
s_3	5	2	4			1
$-z$	-2	-3	1			

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	4	$-\frac{1}{2}$		1	$-\frac{3}{2}$	
x_2	1	$\frac{1}{2}$	1		$\frac{1}{2}$	
s_3	1				-2	1
$-z$	-3	$-\frac{7}{2}$			$-\frac{1}{2}$	

c) The solution is optimal as all NBV possess a negative objective function coefficient. The optimal solution is: $x_1 = 0$, $x_2 = 1$ and $z^* = 3$.

2.3 Transformation into canonical form

Question 54

a)

- Transformation of the objective function:

$$\text{Maximize } 1 \cdot x_1 + 2 \cdot x_2 = -z$$

- Introduction of slack variables:

$$1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = 100$$

$$6 \cdot x_1 + 9 \cdot x_2 + 1 \cdot x_4 = 720$$

$$0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_5 = 60$$

- Domain:

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

b)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	100	1	1	1		
x_4	720	6	9		1	
x_5	60	0	1			1
z	0	1	2			

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	40	1		1		-1
x_4	180	6			1	-9
x_2	60	0	1			1
z	-120	1				-2

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	10			1	$-\frac{1}{6}$	$\frac{1}{2}$
x_1	30	1			$\frac{1}{6}$	$-\frac{3}{2}$
x_2	60		1		0	1
z	-150				$-\frac{1}{6}$	$-\frac{1}{2}$

Question 55

a)

Maximize

$$z = 5 \cdot x_1 - 7 \cdot x_2 + 3 \cdot x_3 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$$

subject to

$$\begin{aligned} 1 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 &= 17 \\ 2 \cdot x_1 + 4 \cdot x_2 - 2 \cdot x_3 + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 &= 11 \\ -1 \cdot x_1 - 1 \cdot x_2 + 3 \cdot x_3 + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot s_3 &= 8 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

b)

BV	Value	x_1	x_2	x_3	s_1	s_2	s_3
s_1	17	1	1	2	1		
s_2	11	2	4	-2		1	
s_3	8	-1	-1	3			1
$-z$	0	5	-7	3			

BV	Value	x_1	x_2	x_3	s_1	s_2	s_3
s_1	$\frac{23}{2}$		-1	3	1	$-\frac{1}{2}$	
x_1	$\frac{11}{2}$	1	2	-1		$\frac{1}{2}$	
s_3	$\frac{27}{2}$		1	2		$\frac{1}{2}$	1
$-z$	$-\frac{55}{2}$		-17	8		$-\frac{5}{2}$	

BV	Value	x_1	x_2	x_3	s_1	s_2	s_3
x_3	$\frac{23}{6}$		$-\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{1}{6}$	
x_1	$\frac{56}{6}$	1	$\frac{5}{3}$		$\frac{1}{3}$	$\frac{2}{6}$	
s_3	$\frac{35}{6}$		$\frac{5}{3}$		$-\frac{2}{3}$	$\frac{5}{6}$	1
$-z$	$-\frac{349}{6}$		$-\frac{43}{3}$		$-\frac{8}{3}$	$-\frac{7}{6}$	

Question 56

a)

Alternative 1

Maximize

$$-z = -2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3^+ - 2 \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 - M \cdot y_1$$

subject to

$$-1 \cdot x_1 - 2 \cdot x_2 - 1 \cdot x_3^+ + 1 \cdot x_3^- + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot y_1 = 9$$

$$-1 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3^+ + 1 \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot y_1 = 0$$

$$0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3^+ - 1 \cdot x_3^- + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot y_1 = 1$$

$$x_1, x_2, x_3^+, x_3^-, s_1, s_2, y_1 \geq 0$$

Another elementary row operation on objective function necessary:

Maximize

$$-z = (-2 - 1 \cdot M) \cdot x_1 + 1 \cdot x_2 + (2 - 1 \cdot M) \cdot x_3^+ + (-2 + 1 \cdot M) \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot y_1$$

subject to

$$-1 \cdot x_1 - 2 \cdot x_2 - 1 \cdot x_3^+ + 1 \cdot x_3^- + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot y_1 = 9$$

$$-1 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3^+ + 1 \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot y_1 = 0$$

$$0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3^+ - 1 \cdot x_3^- + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot y_1 = 1$$

$$x_1, x_2, x_3^+, x_3^-, s_1, s_2, y_1 \geq 0$$

Alternative 2

Maximize

$$-z = -2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3^+ - 2 \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4$$

subject to

$$-1 \cdot x_1 - 2 \cdot x_2 - 1 \cdot x_3^+ + 1 \cdot x_3^- + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 = 9$$

$$-1 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3^+ + 1 \cdot x_3^- + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 = 0$$

$$1 \cdot x_1 - 0 \cdot x_2 + 1 \cdot x_3^+ - 1 \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot s_3 + 0 \cdot s_4 = 0$$

$$0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3^+ - 1 \cdot x_3^- + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 1 \cdot s_4 = 1$$

$$x_1, x_2, x_3^+, x_3^-, s_1, s_2, s_3, s_4 \geq 0$$

Alternative 2 is possible, because the right side of the equality constraint is 0. Instead of adding an artificial variable and using the M-Method, the constraint $-1x_1 + 0x_2 - 1x_3 = 0$ is split into two constraints: $-1x_1 + 0x_2 - 1x_3 \leq 0$ and $-1x_1 + 0x_2 - 1x_3 \geq 0$. The second constraint can be multiplied with (-1) to transform it into a \leq -constraint: $1x_1 - 0x_2 + 1x_3 \leq 0$. Afterwards both constraints can be transformed to canonical form by adding a slack variable.

b)

In the following alternative 2 is used for the simplex algorithm to avoid using the M-Method. Nevertheless, both alternatives are valid and lead to the same optimal solution.

BV	Value	x_1	x_2	x_3^+	x_3^-	s_1	s_2	s_3	s_4
s_1	9	-1	-2	-1	1	1			
s_2	0	-1	0	-1	1		1		
s_3	0	1	0	1	-1			1	
s_4	1	0	1	1	-1				1
z	0	-2	1	2	-2				

BV	Value	x_1	x_2	x_3^+	x_3^-	s_1	s_2	s_3	s_4
s_1	9	0	-2		0	1		1	
s_2	0	0	0		0		1	1	
x_3^+	0	1	0	1	-1			1	
s_4	1	-1	1		0			-1	1
z	0	-4	1		0			-2	

BV	Value	x_1	x_2	x_3^+	x_3^-	s_1	s_2	s_3	s_4
s_1	11	-2			0	1		-1	2
s_2	0	0			0		1	1	0
x_3^+	0	1		1	-1			1	0
x_2	1	-1	1		0			-1	1
z	-1	-3			0			-1	-1

Question 57

a)

Maximize

$$z = 1,000 \cdot x_1 + 3,000 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$$

subject to

$$1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 = 100$$

$$1 \cdot x_1 + 4 \cdot x_2 + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 = 160$$

$$1,000 \cdot x_1 + 2,000 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot s_3 = 110,000$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

b)

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	100	1	1	1		
s_2	160	1	4		1	
s_3	110,000	1,000	2,000			1
$-z$	0	1,000	3,000			

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	60	$\frac{3}{4}$		1	$-\frac{1}{4}$	
x_2	40	$\frac{1}{4}$	1		$\frac{1}{4}$	
s_3	30,000	500			-500	1
$-z$	-120,000	250			-750	

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	15			1	$\frac{1}{2}$	$-\frac{3}{2,000}$
x_2	25		1		$\frac{1}{2}$	$-\frac{1}{2,000}$
x_1	60	1			-1	$\frac{2}{1,000}$
$-z$	-135,000				-500	$-\frac{1}{2}$

Question 58

a)

Maximize

$$z = 3 \cdot x_1 + 5 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

subject to

$$1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot s_1 + 0 \cdot s_2 = 10$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 1 \cdot s_2 = 6$$

$$x_1, x_2, s_1, s_2, \geq 0$$

b)

BV	Value	x_1	x_2	s_1	s_2
s_1	10	1	1	1	
s_2	6	0	1		1
$-z$	0	3	5		

BV	Value	x_1	x_2	s_1	s_2
s_1	4	1		1	-1
x_2	6	0	1		1
$-z$	-30	3			-5

BV	Value	x_1	x_2	s_1	s_2
x_1	4	1		1	-1
x_2	6		1	0	1
$-z$	-42			-3	-2

2.4 Dual Simplex Method / Big-M Method: Basics

Question 59

A feasible and optimal solution can be determined if the linear program is dual feasible and the dual program therefore possesses a canonical form. That is the case if the objective function coefficients of all non-basic variables are smaller or equal to zero.

Question 60

a)

The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

Maximize

$$z = 2 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + -M \cdot y_1$$

subject to

$$\begin{aligned} -2 \cdot x_1 + 5 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot y_1 &= 0 \\ 1 \cdot x_1 + 2 \cdot x_2 + 0 \cdot x_3 - 1 \cdot x_4 + 0 \cdot x_5 + 1 \cdot y_1 &= 9 \\ 2 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 + 0 \cdot y_1 &= 12 \\ x_1, x_2, x_3, x_4, x_5, y_1 &\geq 0 \end{aligned}$$

BV	Value	x_1	x_2	x_3	x_4	x_5	y_1
x_3	0	-2	5	1			
y_1	9	1	2		-1		1
x_5	12	2	1			1	
$-z$	0	2	1				$-M$

BV	Value	x_1	x_2	x_3	x_4	x_5	y_1
x_3	0	-2	5	1			
y_1	9	1	2		-1		1
x_5	12	2	1			1	
$-z$	$9M$	$2 + M$	$1 + 2 \cdot M$		$-M$		

BV	Value	x_1	x_2	x_3	x_4	x_5	y_1
x_2	0	$-\frac{2}{5}$	1	$\frac{1}{5}$			
y_1	9	$\frac{9}{5}$		$-\frac{2}{5}$	-1		1
x_5	12	$\frac{12}{5}$		$-\frac{1}{5}$		1	
$-z$	$9M$	$\frac{12}{5} + \frac{9}{5} \cdot M$		$-\frac{1}{5} - \frac{2}{5} \cdot M$	$-M$		

BV	Value	x_1	x_2	x_3	x_4	x_5	y_1
x_2	2		1	$\frac{1}{9}$	$-\frac{2}{9}$		$\frac{2}{9}$
x_1	5	1		$-\frac{2}{9}$	$-\frac{5}{9}$		$\frac{5}{9}$
x_5	0			$\frac{1}{3}$	$\frac{4}{3}$	1	$-\frac{4}{3}$
$-z$	-12			$\frac{1}{3}$	$\frac{4}{3}$		$-\frac{4}{3} - M$

As no more M is found in the objective function, the M -column can be deleted.

b)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	2		1	$\frac{1}{9}$	$-\frac{2}{9}$	
x_1	5	1		$-\frac{2}{9}$	$-\frac{5}{9}$	
x_5	0			$\frac{1}{3}$	$\frac{4}{3}$	1
$-z$	-12			$\frac{1}{3}$	$\frac{4}{3}$	

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	2		1	$\frac{1}{6}$		$\frac{1}{6}$
x_1	5	1		$-\frac{1}{12}$		$\frac{5}{12}$
x_4	0			$\frac{1}{4}$	1	$\frac{3}{4}$
$-z$	-12			0		-1

Question 61

a)

The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

Maximize

$$z = -1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 - M \cdot y_1$$

subject to

$$\begin{aligned} 1 \cdot x_1 + 3 \cdot x_2 + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot y_1 &= 7 \\ -1 \cdot x_1 - \frac{1}{2} \cdot x_2 + 0 \cdot s_1 - 1 \cdot s_2 + 0 \cdot s_3 + 1 \cdot y_1 &= 1 \\ 2 \cdot x_1 + 2 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot s_3 + 0 \cdot y_1 &= 4 \\ x_1, x_2, s_1, s_2, s_3, y_1 &\geq 0 \end{aligned}$$

b)

BV	Value	x_1	x_2	s_1	s_2	s_3	y_1
s_1	7	1	3	1			
y_1	1	-1	$-\frac{1}{2}$		-1		1
s_3	4	2	2			1	
$-z$	M	$-1 - M$	$1 - \frac{1}{2} \cdot M$		$-M$		

The tableau is optimal, which means all objective function coefficients of the non-basic variables are smaller or equal 0 and at least one artificial variable punished by M is still in the basis. However, the artificial variable is still in the basis and has a value unequal to 0. Therefore, the optimal solution is infeasible and no feasible solution can be found for the LP.

Question 62

a)

Maximize

$$z = 1 \cdot x_1 + 2 \cdot x_2 + 0 \cdot s_1 - M \cdot y_1$$

subject to

$$3 \cdot x_1 - 2 \cdot x_2 + 1 \cdot s_1 + 0 \cdot y_1 = 10$$

$$1 \cdot x_1 - 4 \cdot x_2 + 0 \cdot s_1 + 1 \cdot y_1 = 2$$

$$x_1, x_2, s_1, y_1 \geq 0$$

Another elementary row operation on objective function necessary:

Maximize

$$z + (2 \cdot M) = (1 + M) \cdot x_1 + (2 - 4 \cdot M) \cdot x_2 + 0 \cdot s_1 + 0 \cdot y_1$$

subject to

$$3 \cdot x_1 - 2 \cdot x_2 + 1 \cdot s_1 + 0 \cdot y_1 = 10$$

$$1 \cdot x_1 - 4 \cdot x_2 + 0 \cdot s_1 + 1 \cdot y_1 = 2$$

$$x_1, x_2, s_1, y_1 \geq 0$$

b) The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

BV	Value	x_1	x_2	s_1	y_1
s_1	10	3	-2	1	
y_1	2	1	-4		1
$-z$	$2 \cdot M$	$1 + M$	$2 - 4 \cdot M$		

BV	Value	x_1	x_2	s_1	y_1
s_1	4		10	1	-3
x_1	2	1	-4		1
$-z$	-2		6		$-1 - M$

The last column can be eliminated as the variable punished by M has left the basis.

c)

BV	Value	x_1	x_2	s_1
s_1	4		10	1
x_1	2	1	-4	
$-z$	-2		6	

BV	Value	x_1	x_2	s_1
x_2	$\frac{4}{10}$		1	$\frac{1}{10}$
x_1	$\frac{36}{10}$	1		$\frac{4}{10}$
$-z$	$-\frac{44}{10}$			$-\frac{6}{10}$

Question 63

a)

Maximize

$$-z = \frac{1}{2} \cdot x_1 + \frac{1}{10} \cdot x_2^+ - \frac{1}{10} \cdot x_2^- + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 - \frac{4}{7} - M \cdot y_1$$

subject to

$$1 \cdot x_1 + 1 \cdot x_2^+ - 1 \cdot x_2^- + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot y_1 = 4$$

$$1 \cdot x_1 - 3 \cdot x_2^+ + 3 \cdot x_2^- + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot y_1 = 6$$

$$1 \cdot x_1 + 0 \cdot x_2^+ + 0 \cdot x_2^- + 0 \cdot s_1 + 0 \cdot s_2 - 1 \cdot s_3 + 0 \cdot s_4 + 1 \cdot y_1 = 1$$

$$0 \cdot x_1 + 1 \cdot x_2^+ - 1 \cdot x_2^- + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 1 \cdot s_4 + 0 \cdot y_1 = 1$$

$$x_1, x_2^+, x_2^-, s_1, s_2, s_3, s_4, y_1 \geq 0$$

Another elementary row operation on objective function necessary:

Maximize

$$-z + \frac{4}{7} + M = \left(\frac{1}{2} + M\right) \cdot x_1 + \frac{1}{10} \cdot x_2^+ - \frac{1}{10} \cdot x_2^- + 0 \cdot s_1 + 0 \cdot s_2 - M \cdot s_3 + 0 \cdot s_4 + 0 \cdot y_1$$

subject to

$$1 \cdot x_1 + 1 \cdot x_2^+ - 1 \cdot x_2^- + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot y_1 = 4$$

$$1 \cdot x_1 - 3 \cdot x_2^+ + 3 \cdot x_2^- + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot y_1 = 6$$

$$1 \cdot x_1 + 0 \cdot x_2^+ + 0 \cdot x_2^- + 0 \cdot s_1 + 0 \cdot s_2 - 1 \cdot s_3 + 0 \cdot s_4 + 1 \cdot y_1 = 1$$

$$0 \cdot x_1 + 1 \cdot x_2^+ - 1 \cdot x_2^- + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 1 \cdot s_4 + 0 \cdot y_1 = 1$$

$$x_1, x_2^+, x_2^-, s_1, s_2, s_3, s_4, y_1 \geq 0$$

Additionally, the value $-\frac{4}{7}$ is brought to the z-side of the objective function by adding $\frac{4}{7}$ to the objective function.

- b) The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

BV	Value	x_1	x_2^+	x_2^-	s_1	s_2	s_3	s_4	y_1
s_1	4	1	1	-1	1				
s_2	6	1	-3	3		1			
y_1	1	1						-1	1
s_4	1		1	-1					1
z	$\frac{4}{7} + M$	$\frac{1}{2} + M$	$\frac{1}{10}$	$-\frac{1}{10}$				$-M$	

BV	Value	x_1	x_2^+	x_2^-	s_1	s_2	s_3	s_4	y_1
s_1	3		1	-1	1		1		-1
s_2	5		-3	3		1	1		-1
x_1	1	1					-1		1
s_4	1		1	-1				1	
z	$\frac{1}{14}$		$\frac{1}{10}$	$-\frac{1}{10}$			$\frac{1}{2}$		$-\frac{1}{2} - M$

The last column can be eliminated as the variable punished by M has left the basis.

c)

BV	Value	x_1	x_2^+	x_2^-	s_1	s_2	s_3	s_4
s_3	3		1	-1	1		1	
s_2	2		-4	4	-1	1		
x_1	4	1	1	-1	1			
s_4	1		1	-1	0			1
z	$-\frac{10}{7}$		$-\frac{2}{5}$	$\frac{2}{5}$	$-\frac{1}{2}$			

BV	Value	x_1	x_2^+	x_2^-	s_1	s_2	s_3	s_4
s_3	$\frac{7}{2}$		0		$\frac{3}{4}$	$\frac{1}{4}$	1	
x_2^-	$\frac{1}{2}$		-1	1	$-\frac{1}{4}$	$\frac{1}{4}$		
x_1	$\frac{9}{2}$	1	0		$\frac{3}{4}$	$\frac{1}{4}$		
s_4	$\frac{3}{2}$	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	1
z	$-\frac{57}{35}$	0	0	0	$-\frac{2}{5}$	$-\frac{1}{10}$	0	0

Question 64

a)

Maximize

$$-z = -3 \cdot x_1^+ + 3 \cdot x_1^- + 2 \cdot x_2^+ - 2 \cdot x_2^- - 1 \cdot x_3 + 4 \cdot x_4 - M \cdot y_1 - M \cdot y_2 - M \cdot y_3$$

subject to

$$\begin{aligned} 1 \cdot x_1^+ - 1 \cdot x_1^- + 1 \cdot x_2^+ - 1 \cdot x_2^- - 4 \cdot x_3 + 2 \cdot x_4 - 1 \cdot s_1 + 1 \cdot y_1 &= 4 \\ -3 \cdot x_1^+ + 3 \cdot x_1^- + 1 \cdot x_2^+ - 1 \cdot x_2^- - 2 \cdot x_3 + 0 \cdot x_4 + 1 \cdot s_2 &= 6 \\ 0 \cdot x_1^+ + 0 \cdot x_1^- - 1 \cdot x_2^+ + 1 \cdot x_2^- + 0 \cdot x_3 + 1 \cdot x_4 + 1 \cdot y_2 &= 1 \\ 1 \cdot x_1^+ - 1 \cdot x_1^- + 1 \cdot x_2^+ - 1 \cdot x_2^- - 1 \cdot x_3 + 0 \cdot x_4 + 1 \cdot y_3 &= 0 \end{aligned}$$

$$x_1^+, x_1^-, x_2^+, x_2^-, x_3, x_4, s_1, s_2, y_1, y_2, y_3 \geq 0$$

Another elementary row operation on objective function necessary:

Maximize

$$\begin{aligned} -z + 5M &= (-3 + 2M) \cdot x_1^+ + (3 - 2M) \cdot x_1^- + (2 + M) \cdot x_2^+ + (-2 - M) \cdot x_2^- \\ &\quad + (-1 - 5M) \cdot x_3 + (4 + 3M) \cdot x_4 - M \cdot s_1 + 0 \cdot s_2 + 0 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 \end{aligned}$$

subject to

$$\begin{aligned} 1 \cdot x_1^+ - 1 \cdot x_1^- + 1 \cdot x_2^+ - 1 \cdot x_2^- - 4 \cdot x_3 + 2 \cdot x_4 - 1 \cdot s_1 + 1 \cdot y_1 &= 4 \\ -3 \cdot x_1^+ + 3 \cdot x_1^- + 1 \cdot x_2^+ - 1 \cdot x_2^- - 2 \cdot x_3 + 0 \cdot x_4 + 1 \cdot s_2 &= 6 \\ 0 \cdot x_1^+ + 0 \cdot x_1^- - 1 \cdot x_2^+ + 1 \cdot x_2^- + 0 \cdot x_3 + 1 \cdot x_4 + 1 \cdot y_2 &= 1 \\ 1 \cdot x_1^+ - 1 \cdot x_1^- + 1 \cdot x_2^+ - 1 \cdot x_2^- - 1 \cdot x_3 + 0 \cdot x_4 + 1 \cdot y_3 &= 0 \end{aligned}$$

$$x_1^+, x_1^-, x_2^+, x_2^-, x_3, x_4, s_1, s_2, y_1, y_2, y_3 \geq 0$$

b) The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

BV	Value	x_1^+	x_1^-	x_2^+	x_2^-	x_3	x_4	s_1	s_2	y_1	y_2	y_3
y_1	4	1	-1	1	-1	-4	2	-1		1		
s_2	6	-3	3	1	-1	-2			1			
y_2	1			-1	1		1				1	
y_3	0	1	-1	1	-1	-1						1
z	$5M$	$-3+2M$	$3-2M$	$2+M$	$-2-M$	$-1-5M$	$4+3M$	$-M$				

BV	Value	x_1^+	x_1^-	x_2^+	x_2^-	x_3	x_4	s_1	s_2	y_1	y_2	y_3
y_1	2	1	-1	3	-3	-4		-1		1		
s_2	6	-3	3	1	-1	-2			1			
x_4	1			-1	1		1				1	
y_3	0	1	-1	1	-1	-1						1
z	$-4+2M$	$-3+2M$	$3-2M$	$6+4M$	$-6-4M$	$-1-5M$		$-M$			$-4-3M$	

BV	Value	x_1^+	x_1^-	x_2^+	x_2^-	x_3	x_4	s_1	s_2	y_1	y_3
y_1	2	-2	2			-1		-1		1	
s_2	6	-4	4			-1			1		-1
x_4	1	1	-1			-1	1				1
x_2^+	0	1	-1	1	-1	-1					1
z	$-4+2M$	$-9-2M$	$9+2M$			$5-M$		$-M$			$-6-4M$

BV	Value	x_1^+	x_1^-	x_2^+	x_2^-	x_3	x_4	s_1	s_2	y_1
x_1^-	1	-1	1			$-\frac{1}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$
s_2	2					1		2	1	-2
x_4	2					$-\frac{3}{2}$	1	$-\frac{1}{2}$		$\frac{1}{2}$
x_2^+	1			1	-1	$-\frac{3}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$
z	-13					$\frac{19}{2}$		$\frac{9}{2}$		$-\frac{9}{2}-M$

c)

BV	Value	x_1^+	x_1^-	x_2^+	x_2^-	x_3	x_4	s_1	s_2
x_1^-	1	-1	1			$-\frac{1}{2}$		$-\frac{1}{2}$	
s_2	2					1		2	1
x_4	2					$-\frac{3}{2}$	1	$-\frac{1}{2}$	
x_2^+	1			1	-1	$-\frac{3}{2}$		$-\frac{1}{2}$	
z	-13					$\frac{19}{2}$		$\frac{9}{2}$	

BV	Value	x_1^+	x_1^-	x_2^+	x_2^-	x_3	x_4	s_1	s_2
x_1^-	2	-1	1					$\frac{1}{2}$	$\frac{1}{2}$
x_3	2					1		2	1
x_4	5						1	$\frac{5}{2}$	$\frac{3}{2}$
x_2^+	4			1	-1			$\frac{5}{2}$	$\frac{3}{2}$
z	-32							$-\frac{29}{2}$	$-\frac{19}{2}$

Question 65

a)

Maximize

$$-z = -200 \cdot x_1 - 160 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$$

subject to

$$-6 \cdot x_1 - 2 \cdot x_2 + s_1 = -12$$

$$-2 \cdot x_1 - 2 \cdot x_2 + s_2 = -8$$

$$-4 \cdot x_1 - 12 \cdot x_2 + s_3 = -24$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

This is not the canonical form by definition, because the right-hand sides are negative. However, the LP is dual feasible, because all objective function coefficients are ≤ 0 . Therefore, the dual simplex algorithm can be used if at least one right-hand side is negative as it is the case here. We do not transform the LP further, but use the dual simplex algorithm to solve the LP.

b)

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	-12	-6	-2	1		
s_2	-8	-2	-2		1	
s_3	-24	-4	-12			1
z	0	-200	-160			

BV	Value	x_1	x_2	s_1	s_2	s_3
s_1	-8	$-\frac{16}{3}$		1		$-\frac{1}{6}$
s_2	-4	$-\frac{4}{3}$			1	$-\frac{1}{6}$
x_2	2	$\frac{1}{3}$	1			$-\frac{1}{12}$
z	320	$-\frac{440}{3}$				$-\frac{40}{3}$

BV	Value	x_1	x_2	s_1	s_2	s_3
x_1	$\frac{3}{2}$	1	$-\frac{3}{16}$			$\frac{1}{32}$
s_2	-2		$-\frac{1}{4}$	1		$-\frac{1}{8}$
x_2	$\frac{3}{2}$		1	$\frac{1}{16}$		$-\frac{3}{32}$
z	540			$-\frac{55}{2}$		$-\frac{35}{4}$

BV	Value	x_1	x_2	s_1	s_2	s_3
x_1	1	1		$-\frac{1}{4}$	$\frac{1}{4}$	
s_3	16			2	-8	1
x_2	3		1	$\frac{1}{4}$	$-\frac{3}{4}$	
z	680			-10	-70	

c)

The application of the simplex algorithm is not necessary, because the objective function coefficients of all non-basic variables are smaller or equal to zero.

Question 66

a) Maximize

$$z = 1 \cdot x_1 + 1.5 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 - M \cdot y_1$$

subject to

$$\begin{aligned} 0.2 \cdot x_1 + 0.18 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot y_1 &= 10,000 \\ 0.009 \cdot x_1 + 0.01 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot y_1 &= 2,000 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 - 1 \cdot x_5 + 1 \cdot y_1 &= 8,000 \\ x_1, x_2, x_3, x_4, x_5, y_1 &\geq 0 \end{aligned}$$

Another elementary row operation on objective function necessary:

Maximize

$$z + 8,000 \cdot M = (1 + M) \cdot x_1 + 1.5 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 - M \cdot x_5 + 0 \cdot y_1$$

subject to

$$\begin{aligned} 0.2 \cdot x_1 + 0.18 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot y_1 &= 10,000 \\ 0.009 \cdot x_1 + 0.01 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot y_1 &= 2,000 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 - 1 \cdot x_5 + 1 \cdot y_1 &= 8,000 \\ x_1, x_2, x_3, x_4, x_5, y_1 &\geq 0 \end{aligned}$$

b) The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

BV	Value	x_1	x_2	x_3	x_4	x_5	y_1
x_3	10,000	$\frac{1}{5}$	$\frac{18}{100}$	1			
x_4	2,000	$\frac{9}{1,000}$	$\frac{1}{100}$		1		
y_1	8,000	1	0			-1	1
$-z$	$8,000 \cdot M$	$1 + M$	$\frac{3}{2}$			$-M$	

BV	Value	x_1	x_2	x_3	x_4	x_5	y_1
x_3	8,400		$\frac{18}{100}$	1		$\frac{1}{5}$	$-\frac{1}{5}$
x_4	1,928		$\frac{1}{100}$		1	$\frac{9}{1,000}$	$-\frac{9}{1,000}$
x_1	8,000	1	0			-1	1
$-z$	-8,000		$\frac{3}{2}$			1	$-M - 1$

c)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	$\frac{140,000}{3}$		1	$\frac{50}{9}$		$\frac{10}{9}$
x_4	$\frac{4,384}{3}$			$-\frac{1}{18}$	1	$-\frac{19}{9,000}$
x_1	8,000	1		0		-1
$-z$	-78,000			$-\frac{25}{3}$		$-\frac{2}{3}$

Question 67

a) Maximize

$$z = 5 \cdot x_1 + 2 \cdot x_2$$

subject to

$$50 \cdot x_1 + 4 \cdot x_2 \geq 1,500$$

$$5 \cdot x_1 + \frac{1}{3} \cdot x_2 \leq 450$$

$$x_1 \leq 60$$

$$x_2 \leq 700$$

$$x_1, x_2 \geq 0$$

b) Maximize

$$z = 5 \cdot x_1 + 2 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 - M \cdot y_1$$

subject to

$$\begin{aligned} 50 \cdot x_1 + 4 \cdot x_2 - 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 1 \cdot y_1 &= 1,500 \\ 5 \cdot x_1 + \frac{1}{3} \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot y_1 &= 450 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 + 0 \cdot x_6 + 0 \cdot y_1 &= 60 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 + 0 \cdot y_1 &= 700 \\ x_1, x_2, x_3, x_4, x_5, x_6, y_1 &\geq 0 \end{aligned}$$

Another elementary row operation on objective function necessary:

Maximize

$$z + 1,500 \cdot M = (5 + 50 \cdot M) \cdot x_1 + (2 + 4 \cdot M) \cdot x_2 - M \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot y_1$$

subject to

$$\begin{aligned} 50 \cdot x_1 + 4 \cdot x_2 - 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 1 \cdot y_1 &= 1,500 \\ 5 \cdot x_1 + \frac{1}{3} \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot y_1 &= 450 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 + 0 \cdot x_6 + 0 \cdot y_1 &= 60 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 + 0 \cdot y_1 &= 700 \\ x_1, x_2, x_3, x_4, x_5, x_6, y_1 &\geq 0 \end{aligned}$$

c) The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	y_1
y_1	1,500	50	4	-1				1
x_4	450	5	$\frac{1}{3}$		1			
x_5	60	1				1		
x_6	700		1				1	
$-z$	$1,500 \cdot M$	$5 + 50 \cdot M$	$2 + 4 \cdot M$	$-M$				

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	y_1
x_1	30	1	$\frac{2}{25}$	$-\frac{1}{50}$				$\frac{1}{50}$
x_4	300		$-\frac{1}{15}$	$\frac{1}{10}$	1			$-\frac{1}{10}$
x_5	30		$-\frac{2}{25}$	$\frac{1}{50}$		1		$-\frac{1}{50}$
x_6	700		1				1	
$-z$	-150		$\frac{8}{5}$	$\frac{1}{10}$				$-\frac{1}{10} - M$

Now the last column of the tableau can be eliminated and a feasible basic solution of the linear program has been reached.

d)

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_1	30	1	$\frac{2}{25}$	$-\frac{1}{50}$			
x_4	300		$-\frac{1}{15}$	$\frac{1}{10}$	1		
x_5	30		$-\frac{2}{25}$	$\frac{1}{50}$		1	
x_6	700		1				1
$-z$	-150		$\frac{8}{5}$	$\frac{1}{10}$			

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	375	$\frac{25}{2}$	1	$-\frac{1}{4}$			
x_4	325	$\frac{5}{6}$		$\frac{1}{12}$	1		
x_5	60	1				1	
x_6	325	$-\frac{25}{2}$		$\frac{1}{4}$			1
$-z$	-750	-20		$\frac{1}{2}$			

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	700		1				1
x_4	$\frac{650}{3}$	5			1		$-\frac{1}{3}$
x_5	60	1				1	
x_3	1,300	-50		1			4
$-z$	-1,400	5					-2

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_2	700		1				1
x_1	$\frac{130}{3}$	1			$\frac{1}{5}$		$-\frac{1}{15}$
x_5	$\frac{50}{3}$				$-\frac{1}{5}$	1	$\frac{1}{15}$
x_3	$\frac{10,400}{3}$			1	10		$-\frac{2}{3}$
$-z$	$-\frac{4,850}{3}$				-1		$-\frac{5}{3}$

Question 68

a) Maximize

$$z = \frac{1}{2} \cdot (80,000 - x_1) + 1 \cdot x_2$$

subject to

$$\begin{aligned} \frac{3}{2} \cdot x_1 + 1 \cdot x_2 &\leq 150,000 \\ -\frac{1}{5} \cdot x_1 + a \cdot x_2 &\leq 0 \\ x_1 &\leq 80,000 \\ x_1 &\geq 20,000 \\ x_2 &\leq 70,000 \\ x_2 &\geq 20,000 \\ x_1, x_2 &\geq 0 \end{aligned}$$

b) Maximize

$$z = \frac{1}{2} \cdot (80,000 - x_1) + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 - M \cdot y_1 - M \cdot y_2$$

subject to

$$\begin{aligned} \frac{3}{2} \cdot x_1 + 1 \cdot x_2 + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 150,000 \\ -\frac{1}{5} \cdot x_1 + a \cdot x_2 + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 0 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 80,000 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 - 1 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 1 \cdot y_1 + 0 \cdot y_2 &= 20,000 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 1 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 70,000 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 - 1 \cdot s_6 + 0 \cdot y_1 + 1 \cdot y_2 &= 20,000 \\ x_1, x_2, s_1, s_2, s_3, s_4, s_5, s_6, y_1, y_2 &\geq 0 \end{aligned}$$

Another elementary row operation on objective function necessary:

Maximize

$$z - 40,000 + 40,000 \cdot M = \left(-\frac{1}{2} + M\right) \cdot x_1 + (1 + M) \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 - M \cdot s_4 + 0 \cdot s_5 - M \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2$$

subject to

$$\begin{aligned} \frac{3}{2} \cdot x_1 + 1 \cdot x_2 + 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 150,000 \\ -\frac{1}{5} \cdot x_1 + a \cdot x_2 + 0 \cdot s_1 + 1 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 0 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 80,000 \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 - 1 \cdot s_4 + 0 \cdot s_5 + 0 \cdot s_6 + 1 \cdot y_1 + 0 \cdot y_2 &= 20,000 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 1 \cdot s_5 + 0 \cdot s_6 + 0 \cdot y_1 + 0 \cdot y_2 &= 70,000 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot s_4 + 0 \cdot s_5 - 1 \cdot s_6 + 0 \cdot y_1 + 1 \cdot y_2 &= 20,000 \\ x_1, x_2, s_1, s_2, s_3, s_4, s_5, s_6, y_1, y_2 &\geq 0 \end{aligned}$$

Additionally, the value $\frac{1}{2} \cdot 80,000$ is brought to the z-side of the objective function by subtracting 40,000 from the objective function.

c)

- (1) As at least 20,000 liters of fruit juice have to be produced and a maximum of $\frac{80,000}{5} = 16,000$ liters of apple juice is available, a feasible solution for the LP does not exist for $\frac{16,000}{20,000} = 0.8 < a \leq 1$.

(2) There is no a for which the LP would be unbound, because the solution space of the LP is already bound through the constraints $x_1 \leq 80,000$ and $x_2 \leq 70,000$, which are independent of a .

(3) As no more than 70,000 liters of fruit juice can be sold and at least Liter $\frac{20,000}{5} = 4,000$ liters of apple juice are produced, the constraint containing a would be redundant for $0 \leq a < 0.057$.

d.1) The dual simplex algorithm cannot be used, because the LP is not dual feasible (not all objective function coefficients ≤ 0). Therefore, the M-Method is used to find a basic feasible solution.

Note: $a = 10\% = 0.1$

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	y_1	y_2
s_1	150,000	$\frac{3}{2}$	1	1							
s_2	0	$-\frac{1}{5}$	$\frac{1}{10}$		1						
s_3	80,000	1				1					
y_1	20,000	1					-1			1	
s_5	70,000		1					1			
y_2	20,000		1						-1		1
$-z$	$-40,000 + 40,000 \cdot M$	$-\frac{1}{2} + M$	$1 + M$				$-M$		$-M$		

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	y_1	y_2
s_1	150,000	$\frac{7}{2}$		1	-10						
x_2	0	-2	1		10						
s_3	80,000	1				1					
y_1	20,000	1					-1			1	
s_5	70,000	2			-10			1			
y_2	20,000	2			-10				-1		1
$-z$	$-40,000 + 40,000 \cdot M$	$\frac{3}{2} + 3 \cdot M$			$-10 - 10 \cdot M$		$-M$		$-M$		

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	y_1	y_2
s_1	115,000			1	$\frac{15}{2}$				$\frac{7}{4}$		$-\frac{7}{4}$
x_2	20,000		1		0				-1		1
s_3	70,000				5	1			$\frac{1}{2}$		$-\frac{1}{2}$
y_1	10,000				5		-1		$\frac{1}{2}$	1	$-\frac{1}{2}$
s_5	50,000							1	1		-1
x_1	10,000	1			-5				$-\frac{1}{2}$		$\frac{1}{2}$
$-z$	$-55,000 + 10,000 \cdot M$				$-\frac{5}{2} + 5 \cdot M$		$-M$		$\frac{3}{4} + \frac{1}{2} \cdot M$		$-\frac{3}{4} - \frac{3}{2} \cdot M$

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	y_1	y_2
s_1	100,000			1			$\frac{3}{2}$		1	$-\frac{3}{2}$	-1
x_2	20,000		1						-1		1
s_3	60,000					1	1			-1	
s_2	2,000				1		$-\frac{1}{5}$		$\frac{1}{10}$	$\frac{1}{5}$	$-\frac{1}{10}$
s_5	50,000							1	1		-1
x_1	20,000	1					-1			1	
$-z$	-50,000						$-\frac{1}{2}$		1	$\frac{1}{2} - 1 \cdot M$	$-1 - 1 \cdot M$

d.2)

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	80,000			1	-10		$\frac{7}{2}$		
x_2	40,000		1		10		-2		
s_3	60,000				0	1	1		
s_6	20,000				10		-2		1
s_5	30,000				-10		2	1	
x_1	20,000	1			0		-1		
$-z$	-70,000				-10		$\frac{3}{2}$		

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	27,500			1	$\frac{15}{2}$			$-\frac{7}{4}$	
x_2	70,000		1		0			1	
s_3	45,000				5	1		$-\frac{1}{2}$	
s_6	50,000				0			1	1
s_4	15,000				-5		1	$\frac{1}{2}$	
x_1	35,000	1			-5			$\frac{1}{2}$	
$-z$	-92,500				$-\frac{5}{2}$			$-\frac{3}{4}$	

2.5 Simplex Algorithm: Special Cases

Question 69

Special cases:

- no feasible solution
- unbounded solution
- multiple optimal solutions
- redundant constraint
- primal degeneracy

For detailed explanations refer to script.

Question 70

- a) (1) Multiple optimal solutions as NBV x_4 possesses objective function coefficient zero.
- (2) Degenerate solution as x_4 possesses value zero as a basic variable.
- (3) Redundant constraint, because all solutions satisfying constraint 2 ($6 \cdot x_1 + 5 \cdot x_2 \leq 80$) will also satisfy constraint 1 ($6 \cdot x_1 + 5 \cdot x_2 \leq 60$).
- (4) Unbound solution space as all coefficients in the pivot column (x_2) possess a value of smaller or equal to zero.
- (5) No feasible start form for dual or primal simplex. Introduction of Big M and artificial variable to generate canonical form. Artificial variable does not leave the basis, therefore no feasible solution to the problem exists.
- b) (1) The simplex algorithm stops when reaching the first optimal solution.
- (2) A pivot step without changing the objective function value is conducted.

Hint:

There exists a possibility that the simplex algorithm does not leave the degenerate solution and therefore does not reach the optimal solution.

- (3) No influence on simplex algorithm.
- (4) Due to the negative coefficients in the pivot column no more pivot steps can be conducted. The simplex algorithm terminates with evidence for an unbounded solution space.
- (5) Canonical start form is technically optimal (i.e. all coefficients in objective function row ≤ 0). The Simplex algorithm terminates, however, the artificial variable is still in the basis.

Question 71

- a) $\alpha \geq 0$
- b) $\alpha > 0$.
- c) α cannot be set such that the LP has just one optimal solution which is degenerate.
- d) $\alpha \geq 0$.

- e) α cannot be set such that the LP has an unbounded feasible region.

Question 72

- a) $\beta \geq 0$.
 b) β cannot be set such that the LP has at least one optimal solution which is not degenerate.
 c) β cannot be set such that the LP has just one optimal solution which is degenerate.
 d) $\beta = 0$.
 e) $\beta > 0$.

2.6 Sensitivity Analysis: Shadow Price and Reduced Cost

Question 73

- a)
- The first production factor (plantable area) possesses a shadow price of zero meaning one more ha of plantable area has no value to the farmer. The reason for that is, that not all of the available area is used in the optimal solution ($x_3 = 15$).
 - The second production factor (working days) possesses a shadow price of 500 meaning one more available working day would increase the objective function value by 500 Euro.
 - The third production factor (budget) possesses a shadow price of $\frac{1}{2}$ meaning increasing the budget by one would increase the objective function value by 50 cent.

b)

- For each additional unit of asparagus grown by the farmer there are 3 ha of plantable area ($x_3 = 3$), 4 working days ($x_4 = 4$) and 2,000 Euro of budget ($x_5 = 2,000$) less available for growing potatoes and cereal crops.

On the one hand, this reduces the objective function value by

$$\sum_{i=3}^5 x_i \cdot \text{ShadowPrice}_i = 0 \cdot 3 + 500 \cdot 4 + \frac{1}{2} \cdot 2,000 = 3,000$$

On the other hand, the farmer gains a profit margin of 4,000 Euro for every unit of asparagus he grows.

The reduced cost (difference between profit margin and opportunity costs: $4,000 - 3,000 = 1,000$) is positive. It would therefore be profitable for the farmer to add the product “asparagus“ to the production plan.

- For every additional unit of corn grown by the farmer there are 3 ha of plantable area, 2 working days and 2,000 Euro of budget less available for growing potatoes and cereal crops.

On the one hand, this reduces the objective function value by

$$\sum_{i=3}^5 x_i \cdot \text{ShadowPrice}_i = 0 \cdot 3 + 500 \cdot 2 + \frac{1}{2} \cdot 2,000 = 2,000$$

On the other hand, the farmer gains a profit margin of 2,500 Euro for every unit of corn he grows.

The reduced cost ($2,500 - 2,000 = 500$ Euro) is positive. It would therefore be profitable for the farmer to add the product “corn“ to the production plan.

Finally the farmer will choose the product with the higher reduced costs (asparagus) as he can only choose one additional product for his production plan.

c)

Initially the product “asparagus“ is added to the production plan and an optimal solution is determined using the Simplex Algorithm.

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	100	1	1	3	1		
x_5	160	1	4	4		1	
x_6	110,000	1,000	2,000	2,000			1
-DB	0	1,000	3,000	4,000			

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_3	30	$\frac{3}{8}$		1	$\frac{1}{2}$	$-\frac{1}{8}$	
x_2	10	$-\frac{1}{8}$	1		$-\frac{1}{2}$	$\frac{3}{8}$	
x_6	30,000	500				-500	1
-DB	-150,000	-125			-500	-625	

Now the reduced cost of the product “corn“ is determined.

$$2,500 - \sum_{i=4}^6 x_i \cdot \text{ShadowPrice}_i = 2,500 - (500 \cdot 3 + 625 \cdot 2 + 0 \cdot 2,000) = -250$$

It is not profitable to add the product “corn“ to the production plan, because the reduced cost is negative.

Note: Reduced cost can only be used to decide about adding one more product to the production plan.

Question 74

- The shadow price of producing one more unit of chocolate is 1. The company should therefore pay a maximum of 1 for one additional unit of chocolate capacity.
- For the resources used for the production of one “Calorie Bomb“ which are no longer available for the previous production plan there is a decrease in objective function value by $4 \cdot (+1) + 3 \cdot (+1) = 7$. The contribution margin of the “Calorie Bomb“ hence has to be at least 7 for it to be produced without loss.

2.7 Sensitivity Analysis: Variation of the Objective Function Coefficient

Question 75

a)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	7	1	1	2	1	
x_5	10	2	1	3		1
-CM	0	8	6	$5+\Delta$		

The objective function coefficients are only changed through adding to and subtracting from them in the pivot steps. The optimal tableau is therefore:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4		1	1	2	-1
x_1	3	1		1	-1	1
-CM	-48			$-9+\Delta$	-4	-2

For x_3 to enter the base, it must hold: $-9 + \Delta > 0 \Leftrightarrow \Delta > 9$.

“Export“ would be added to the production plan if its contribution margin was greater 14.

b)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_4	7	1	1	2	1	
x_5	10	2	1	3		1
-CM	0	$8+\Delta$	6	5		

The objective function coefficient $8+\Delta$ is only changed through adding to and subtracting from it in the pivot steps. Following tableau is therefore obtained:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4		1	1	2	-1
x_1	3	1		1	-1	1
-CM	-48	Δ		-9	-4	-2

The objective function coefficient of one base variable is not zero which makes the tableau not optimal. Following tableau is obtained after one more iteration:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4		1	1	2	-1
x_1	3	1		1	-1	1
-CM	$-48 - 3 \cdot \Delta$			$-9 - \Delta$	$-4 + \Delta$	$-2 - \Delta$

For the optimal production plan to remain unchanged, it must hold:

$$-9 - \Delta \leq 0 \Leftrightarrow \Delta \geq -9$$

$$-4 + \Delta \leq 0 \Leftrightarrow \Delta \leq 4$$

$$-2 - \Delta \leq 0 \Leftrightarrow \Delta \geq -2$$

$$\Rightarrow -2 \leq \Delta \leq 4$$

The optimal production plan does not change for a contribution margin of “Wheat Beer“ between $[6,12]$.

Question 76

Maximize

$$-1 \cdot x_1 + 1 \cdot x_2 = z$$

subject to

$$1 \cdot x_1 + 3 \cdot x_2 \leq 7$$

$$1 \cdot x_1 + 2 \cdot x_2 \geq 1$$

$$2 \cdot x_1 + 2 \cdot x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Optimal Tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	1	-2	0	1	0	$-\frac{3}{2}$
x_2	2	1	1	0	0	$\frac{1}{2}$
x_4	3	1	0	0	1	1
$-z$	-2	-2	0	0	0	$-\frac{1}{2}$

a)

New objective function:

Maximize

$$(-1 + \Delta) \cdot x_1 + 1 \cdot x_2 = z$$

The objective function coefficient $-1 + \Delta$ is only changed through adding to and subtracting from it in the pivot steps. The optimal tableau is therefore:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	1	-2	0	1	0	$-\frac{3}{2}$
x_2	2	1	1	0	0	$\frac{1}{2}$
x_4	3	1	0	0	1	1
$-z$	-2	$-2 + \Delta$	0	0	0	$-\frac{1}{2}$

For x_1 not to enter the base, it must hold: $-2 + \Delta \leq 0 \Leftrightarrow \Delta \leq 2$.

For the objective function coefficient of x_1 having a value in the interval $] -\infty, 1]$ the optimal basis does not change.

b)

New objective function:

Maximize

$$-1 \cdot x_1 + (1 + \Delta) \cdot x_2 = z$$

The objective function coefficient $1 + \Delta$ is only changed through adding to and subtracting from it in the pivot steps. The optimal tableau is therefore:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	1	-2	0	1	0	$-\frac{3}{2}$
x_2	2	1	1	0	0	$\frac{1}{2}$
x_4	3	1	0	0	1	1
$-z$	-2	-2	Δ	0	0	$-\frac{1}{2}$

The objective function coefficient of one base variable is not zero which makes the tableau not optimal. Following tableau is obtained after one more iteration:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	1	-2	0	1	0	$-\frac{3}{2}$
x_2	2	1	1	0	0	$\frac{1}{2}$
x_4	3	1	0	0	1	1
$-z$	$-2 - 2 \cdot \Delta$	$-2 - \Delta$	0	0	0	$-\frac{1}{2} - \frac{1}{2} \cdot \Delta$

For the optimal base not to change, it must hold:

$$-2 - \Delta \leq 0 \Leftrightarrow \Delta \geq -2$$

$$-\frac{1}{2} - \frac{1}{2} \cdot \Delta \leq 0 \Leftrightarrow \Delta \geq -1$$

$$\Rightarrow -1 \leq \Delta \leq \infty$$

The optimal base does not change for the objective coefficient of x_2 having a value in the interval $[0, \infty[$.

c)

Use Pricing Out Approach:

$$\alpha - (0 \cdot 4 + 0 \cdot 1 + \frac{1}{2} \cdot 5) > 0$$

$$\alpha - \frac{5}{2} > 0$$

$$\alpha > \frac{5}{2}$$

Solving the new linear program with the simplex algorithm yields the same solution:

Maximize

$$z = -1 \cdot x_1 + 1 \cdot x_2 + \alpha \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 - M \cdot y_1$$

subject to

$$1 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot y_1 = 7$$

$$1 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 - 1 \cdot x_5 + 0 \cdot x_6 + 1 \cdot y_1 = 1$$

$$2 \cdot x_1 + 2 \cdot x_2 + 5 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 + 0 \cdot y_1 = 4$$

$$x_1, x_2, x_3, x_4, x_5, x_6, y_1 \geq 0$$

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	y_1
x_4	7	1	3	4	1			
y_1	1	1	2	1		-1		1
x_6	4	2	2	5			1	
$-z$	M	$-1 + M$	$1 + 2 \cdot M$	$\alpha + M$		$-M$		

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6	y_1
x_4	$\frac{11}{2}$	$-\frac{1}{2}$	0	$\frac{7}{2}$	1	$\frac{3}{2}$		$-\frac{3}{2}$
x_2	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$
x_6	3	1	0	4		1	1	-1
$-z$	$-\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2} + \alpha$		$\frac{1}{2}$		$-\frac{1}{2} - M$

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	1	-2		$-\frac{5}{2}$	1	0	$-\frac{3}{2}$
x_2	2	1	1	$\frac{5}{2}$		0	$\frac{1}{2}$
x_5	3	1		4		1	1
$-z$	-2	-2		$-\frac{5}{2} + \alpha$		0	$-\frac{1}{2}$

For every $\alpha > \frac{5}{2}$ x_3 is in the optimal basis.

Question 77

a)

Maximize

$$z = 0.70 \cdot x_1 + 0.40 \cdot x_2 \quad (1)$$

subject to

$$0.70 \cdot x_1 + 0.80 \cdot x_2 \leq 35 \quad (2)$$

$$0.25 \cdot x_1 + 0.15 \cdot x_2 \leq 10 \quad (3)$$

$$1.50 \cdot x_1 + 0.50 \cdot x_2 \leq 40 \quad (4)$$

$$2.70 \cdot x_1 + 1.80 \cdot x_2 \leq 90 \quad (5)$$

$$3 \cdot x_1 \leq 75 \quad (6)$$

$$5 \cdot x_2 \leq 200 \quad (7)$$

$$x_1, x_2 \geq 0 \quad (8)$$

- (1) The contribution margin is maximized.
- (2) There are in total 35 litres of red wine available.
- (3) There are in total 10 litres of water available.
- (4) There are in total 40 lemons available.
- (5) There are at most 90 minutes available to mix the components.
- (6) There are in total 75 gram cinnamon available.
- (7) There are in total 200 gram clove available.
- (8) The decision variables are non-negative.

x_1 : litres of cinnamon punch to be produced

x_2 : litres of clove punch to be produced

b)

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	35	$\frac{7}{10}$	$\frac{8}{10}$	1	0	0	0	0	0
s_2	10	$\frac{5}{20}$	$\frac{3}{20}$	0	1	0	0	0	0
s_3	40	$\frac{3}{2}$	$\frac{1}{2}$	0	0	1	0	0	0
s_4	90	$\frac{27}{10}$	$\frac{18}{10}$	0	0	0	1	0	0
s_5	75	3	0	0	0	0	0	1	0
s_6	200	0	5	0	0	0	0	0	1
$-z$	0	$\frac{7}{10}$	$\frac{4}{10}$	0	0	0	0	0	0

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	$\frac{35}{2}$	0	$\frac{8}{10}$	1	0	0	0	$-\frac{7}{30}$	0
s_2	$\frac{15}{4}$	0	$\frac{3}{20}$	0	1	0	0	$-\frac{5}{60}$	0
s_3	$\frac{5}{2}$	0	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	0
s_4	$\frac{45}{2}$	0	$\frac{18}{10}$	0	0	0	1	$-\frac{9}{10}$	0
x_1	25	1	0	0	0	0	0	$\frac{1}{3}$	0
s_6	200	0	5	0	0	0	0	0	1
$-z$	$-\frac{35}{2}$	0	$\frac{4}{10}$	0	0	0	0	$-\frac{7}{30}$	0

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	$\frac{27}{2}$	0	0	1	0	$-\frac{16}{10}$	0	$\frac{34}{60}$	0
s_2	3	0	0	0	1	$-\frac{1}{3}$	0	$\frac{4}{60}$	0
x_2	5	0	1	0	0	2	0	-1	0
s_4	$\frac{27}{2}$	0	0	0	0	$-\frac{36}{10}$	1	$\frac{9}{10}$	0
x_1	25	1	0	0	0	0	0	$\frac{1}{3}$	0
s_6	175	0	0	0	0	-10	0	5	1
$-z$	$-\frac{39}{2}$	0	0	0	0	$-\frac{8}{10}$	0	$\frac{1}{6}$	0

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	5	0	0	1	0	$\frac{4}{6}$	$-\frac{34}{54}$	0	0
s_2	2	0	0	0	1	$-\frac{1}{30}$	$-\frac{4}{54}$	0	0
x_2	20	0	1	0	0	-2	$\frac{10}{9}$	0	0
s_5	15	0	0	0	0	-4	$\frac{10}{9}$	1	0
x_1	20	1	0	0	0	$\frac{4}{3}$	$-\frac{20}{54}$	0	0
s_6	100	0	0	0	0	10	$-\frac{50}{9}$	0	1
$-z$	-22	0	0	0	0	$-\frac{4}{30}$	$-\frac{10}{54}$	0	0

c)

The objective function value would decrease by $\frac{4}{30}$ Euro for every lemon less available which is the shadow price of the related constraint.

d)

The shadow price of the related constraint is $\frac{10}{54}$ which means the objective function value increases by $\frac{10}{54}$ for every additional minute of working time. The maximum hourly salary should therefore be 11.11 Euro.

When increasing the available working time further at some point one of the base variables will

have the value zero and leave the basis. The variable s_4 will then enter the basis. The shadow price of the available working time will be equal to zero meaning increasing the available working time holds no value.

The first base variable to leave the basis is s_5 . This is the case as soon as s_4 is increased by more than 13.5 minutes.

e)

The reduced cost of the new punch amount to:

$$\begin{aligned} RC &= 0.30 - \left[0.75 \cdot 0 + 0.2 \cdot 0 + 1 \cdot \frac{4}{30} + 1 \cdot \frac{10}{54} \right] \\ &= -\frac{1}{54} \end{aligned}$$

The reduced cost is smaller than zero and therefore it is not profitable to add the new punch to the range of products.

f) Varying the contribution return of cinnamon punch by a value Δ leads to following changes in the optimal tableau of part b):

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	5	0	0	1	0	$\frac{4}{6}$	$-\frac{34}{54}$	0	0
s_2	2	0	0	0	1	$-\frac{1}{30}$	$-\frac{4}{54}$	0	0
x_2	20	0	1	0	0	-2	$\frac{10}{9}$	0	0
s_5	15	0	0	0	0	-4	$\frac{10}{9}$	1	0
x_1	20	1	0	0	0	$\frac{4}{3}$	$-\frac{20}{54}$	0	0
s_6	100	0	0	0	0	10	$-\frac{50}{9}$	0	1
$-z$	-22	Δ	0	0	0	$-\frac{4}{30}$	$-\frac{10}{54}$	0	0

The tableau is no longer optimal, because the coefficient of the base variable x_1 in the objective function row is not equal to zero. The following tableau is the result of another iteration step of the Simplex Algorithm:

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	5	0	0	1	0	$\frac{4}{6}$	$-\frac{34}{54}$	0	0
s_2	2	0	0	0	1	$-\frac{1}{30}$	$-\frac{4}{54}$	0	0
x_2	20	0	1	0	0	-2	$\frac{10}{9}$	0	0
s_5	15	0	0	0	0	-4	$\frac{10}{9}$	1	0
x_1	20	1	0	0	0	$\frac{4}{3}$	$-\frac{20}{54}$	0	0
s_6	100	0	0	0	0	10	$-\frac{50}{9}$	0	1
$-z$	$-22 - 20 \cdot \Delta$	0	0	0	0	$-\frac{4}{30} - \frac{4}{3} \cdot \Delta$	$-\frac{10}{54} + \frac{20}{54} \cdot \Delta$	0	0

For the basis conducted in part b) to stay optimal, the objective function row coefficients of the non-basic variables have to be smaller or equal to zero. In Conclusion:

$$\Delta \geq -\frac{1}{10} \quad \text{and} \quad \Delta \leq \frac{1}{2}$$

\Rightarrow The contribution return of one litre of Cinnamon punch can fluctuate within the interval $[\frac{6}{10}, \frac{12}{10}]$.

g) Varying the contribution return per lemon by a value Δ leads to following changes in the optimal tableau of part b):

BV	Value	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6
s_1	5	0	0	1	0	$\frac{4}{6}$	$-\frac{34}{54}$	0	0
s_2	2	0	0	0	1	$-\frac{1}{30}$	$-\frac{4}{54}$	0	0
x_2	20	0	1	0	0	-2	$\frac{10}{9}$	0	0
s_5	15	0	0	0	0	-4	$\frac{10}{9}$	1	0
x_1	20	1	0	0	0	$\frac{4}{3}$	$-\frac{20}{54}$	0	0
s_6	100	0	0	0	0	10	$-\frac{50}{9}$	0	1
$-z$	-22	0	0	0	0	$-\frac{4}{30} + \Delta$	$-\frac{10}{54}$	0	0

For the basis conducted in part b) not to be optimal, one of the objective function row coefficients of a non-basic variable has to be greater than zero. In Conclusion: $\Delta > \frac{4}{30}$ meaning the contribution return per lemon has to be greater than $\frac{4}{30}$.

2.8 Dual programs: Basics

Question 78

- The number of iterations of the Simplex Algorithm conducted until an optimal solution is found amounts to about 1.5 to 2 times the number of constraints of the LP that is solved.
- The number of variables of a problem is equal to the number of constraints of its dual problem.
- The objective function values of the optimal solution of the primal and the dual problem are equal (strong duality property). This means it does not make a difference, which of the two problems is solved.
- It is in general more efficient to solve the dual problem with the Simplex Algorithm if it holds:

Number of variables of the problem (= Number of constraints of dual problem) is smaller than number of constraints of the primal problem.

Question 79

Minimize

$$v = 10 \cdot y_1 + 8 \cdot y_2 + 4 \cdot y_3$$

subject to

$$\begin{aligned} 1 \cdot y_1 + 2 \cdot y_2 + 0 \cdot y_3 &\geq 3 \\ 2 \cdot y_1 + \frac{3}{2} \cdot y_2 + 1 \cdot y_3 &\leq 2 \\ y_1 &\geq 0 \\ y_2 &\geq 0 \\ y_3 &\leq 0 \end{aligned}$$

Question 80

Minimize

$$v = 0 \cdot y_1 + 9 \cdot y_2 + 12 \cdot y_3$$

subject to

$$\begin{aligned}2 \cdot y_1 + 1 \cdot y_2 + 2 \cdot y_3 &\geq 2 \\-5 \cdot y_1 + 2 \cdot y_2 + 1 \cdot y_3 &= 1 \\y_1 &\in \mathbb{R} \\y_2 &\leq 0 \\y_3 &\geq 0\end{aligned}$$

2.9 Dual programs: Properties

Question 81

- Strong duality property
- Weak duality property
- Unboundness property
- Complementary slackness

For Explanations refer to script.

Question 82

a)

Minimize

$$v = 160 \cdot y_1 + 110,000 \cdot y_2 + 100 \cdot y_3$$

subject to

$$\begin{aligned}y_1 + 1,000 \cdot y_2 + y_3 &\geq 1,000 \\4 \cdot y_1 + 2,000 \cdot y_2 + y_3 &\geq 3,000 \\y_1, y_2, y_3 &\geq 0\end{aligned}$$

b)

It holds for the optimal tableau that: The reduced cost for the slack variables (= shadow prices) of the primal problem are equal to the optimal values of the decision variables of the dual problem.

The optimal solution of the dual problem is given by $y_1 = 500$ and $y_2 = \frac{1}{2}$ and $v = 135,000$

c)

Taking into consideration the strong and the weak duality property it holds for all feasible solutions of the dual problem: $v \geq 135,000$

Question 83

a) Maximize

$$12 \cdot y_1 + 8 \cdot y_2 + 24 \cdot y_3 = v$$

subject to

$$\begin{aligned}6 \cdot y_1 + 2 \cdot y_2 + 4 \cdot y_3 &\leq 200 \\2 \cdot y_1 + 2 \cdot y_2 + 12 \cdot y_3 &\leq 160 \\y_1 &\geq 0 \\y_2 &\geq 0 \\y_3 &\geq 0\end{aligned}$$

b)

BV	Value	y_1	y_2	y_3	s_1	s_2
s_1	200	6	2	4	1	
s_2	160	2	2	12		1
$-v$	0	12	8	24		

BV	Value	y_1	y_2	y_3	s_1	s_2
s_1	$\frac{440}{3}$	$\frac{16}{3}$	$\frac{4}{3}$		1	$-\frac{1}{3}$
y_3	$\frac{40}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1		$\frac{1}{12}$
$-v$	-320	8	4			-2

BV	Value	y_1	y_2	y_3	s_1	s_2
y_1	$\frac{55}{2}$	1	$\frac{1}{4}$		$\frac{3}{16}$	$-\frac{1}{16}$
y_3	$\frac{35}{4}$		$\frac{1}{8}$	1	$-\frac{1}{32}$	$\frac{3}{32}$
$-v$	-540		2		$-\frac{3}{2}$	$-\frac{3}{2}$

BV	Value	y_1	y_2	y_3	s_1	s_2
y_1	10	1		-2	$\frac{1}{4}$	$-\frac{1}{4}$
y_2	70		1	8	$-\frac{1}{4}$	$\frac{3}{4}$
$-v$	-680			-16	-1	-3

- c) – Both optimal tableaus contain the same values with sometimes differing signs and therefore contain identical information.
- The shadow prices of the primal problem are equal to the values of the optimal solution of the dual problem and vice versa.
- Strong duality property: $v = z$
- Complementary Slackness:
 Both of the decision variables x_1 and x_2 of the primal problem are in the basis. Therefore the related constraints of the dual problem have zero slack: $s_1 = 0$ and $s_2 = 0$.
 The decision variable y_3 of the dual problem is not part of the basis meaning the related constraint of the primal problem has a non-zero slack: $s_3 = 16$.

3 Integer and Mixed-Integer Programming

3.1 Branch-and-Bound: Basics

Question 84

An LP-relaxation of an integer or a mixed-integer linear program (ILP) is obtained by relaxing the integrality constraint. If the ILP is a maximization (minimization) problem, then the objective function value of the optimal solution of the LP-relaxation represents an upper (lower) bound of the objective function value of the optimal solution of the ILP.

Question 85

refer to script

3.2 Branch-and-Bound with Simplex Algorithm

Question 86

K	P_{i^*}	x_1	x_2	z	\bar{z}	\underline{z}	Δ	*)	Branch	UB
$\langle P_a \rangle$	P_a	$\frac{5}{3}$	3	17	17	0	100%	(4)	$x_1 \leq 1 \rightarrow P_b$ $x_1 \geq 2 \rightarrow P_e$	16.6 16.4
$\langle P_b, P_e \rangle$	P_b	1	3.4	16.6	16.6	0	100%	(4)	$x_2 \leq 3 \rightarrow P_d$ $x_2 \geq 4 \rightarrow P_c$	15 16
$\langle P_e, P_c, P_d \rangle$	P_e	2	2.6	16.4	16.4	0	100%	(4)	$x_2 \leq 2 \rightarrow P_f$ $x_2 \geq 3 \rightarrow P_g$	15.5 n. feas.
$\langle P_c, P_f, P_d, P_g \rangle$	P_c	0	4	16	16	16	0%	(1)		

The procedure stops, because $\Delta = 0$, with an optimal solution $x_1 = 0, x_2 = 4$ and the optimal objective function value of 16.

The procedure does not terminate early, because the next found integer solution is already optimal.

Question 87

a)

K	P_{i^*}	x_1	x_2	z	\bar{z}	\underline{z}	Δ	*)	Branch	LB
$\langle P_a \rangle$	P_a	6	2.33	23.83		23.83		(4)	$x_2 \leq 2 \rightarrow P_e$ $x_2 \geq 3 \rightarrow P_b$	24.5 24.35
$\langle P_b, P_e \rangle$	P_b	5.6	3	24.35		24.35		(4)	$x_1 \leq 5 \rightarrow P_c$ $x_1 \geq 6 \rightarrow P_d$	25.21 25.5
$\langle P_e, P_c, P_d \rangle$	P_e	6.5	2	24.5		24.5		(4)	$x_1 \leq 6 \rightarrow P_f$ $x_1 \geq 7 \rightarrow P_g$	n. feas. 25.17
$\langle P_g, P_c, P_d, P_f \rangle$	P_g	7	1.67	25.17		25.17		(4)	$x_2 \leq 1 \rightarrow P_i$ $x_2 \geq 2 \rightarrow P_h$	26.5 26
$\langle P_c, P_d, P_h, P_i, P_f \rangle$	P_c	5	4.08	25,21		25,21		(4)	$x_2 \leq 4 \rightarrow P_j$ $x_2 \geq 5 \rightarrow P_l$	n. feas. 25.91
$\langle P_d, P_l, P_h, P_i, P_f, P_j \rangle$	P_d	6	3	25.5	25.5	25.5	0%	(1)		
$\langle P_l, P_h, P_i, P_f, P_j \rangle$	P_l	4.47	5	25.91				(2)		
$\langle P_h, P_i, P_f, P_j \rangle$	P_h	7	2	26				(1)		
$\langle P_i, P_f, P_j \rangle$	P_i	8	1	26.5				(1)		
$\langle P_f, P_j \rangle$	P_f	–	–	–				(3)		
$\langle P_j \rangle$	P_j	–	–	–				(3)		
$\langle \rangle$										
Opt. sol.		6	3	25.5						

The procedure stops, because the candidate list is empty, with an optimal solution $x_1 = 6$, $x_2 = 3$ and an optimal objective function value of 25.5. Also note that the procedure may be terminated once an optimality gap of $\Delta = 0$ has been determined. [After evaluation of P_d]

- b) The procedure stops after eliminating P_d from the candidate list. The deviation of the objective function value of the best known feasible solution from the objective function value of the optimal solution is with $\Delta = \frac{\bar{z} - \underline{z}}{\underline{z}} = \frac{25.5 - 25.5}{25.5} = 0\%$.

3.3 Branch-and-Bound without Simplex Algorithm

Question 88

a) 1. Determination of a lower bound \underline{z}

- Determine the value per cost unit of the jewellery.

Jewellery	Value	Cost	Value per Cost
Necklace	4	7	$\frac{4}{7}$
Ring	2	2	1
Bracelet	3	4	$\frac{3}{4}$

- Select the jewellery in the order of decreasing value per cost while staying within the budget.

Jewellery	Percentage	cum. Value	cum. Cost
Ring	100%	2	2
Bracelet	100%	5	6
Necklace	0%	5	6

- We get a feasible solution $x_2 = x_3 = 1$ and $x_1 = 0$ with $\underline{z} = 5$

2. Determination of an upper bound \bar{z}

- Select (part of) the jewellery in order of the value per cost while using all of the available budget.

Jewellery	Percentage	cum. Value	cum. Cost
Ring	100%	2	2
Bracelet	100%	5	6
Necklace	57.14%	7.286	10

- We get an infeasible solution $x_2 = x_3 = 1$, $x_1 = \frac{4}{7}$ with $\bar{z} = 7\frac{2}{7}$

b)

K	P_{i^*}	x_1	x_2	x_3	z	\bar{z}	\underline{z}	Δ	*)	Branch	UB
$\langle P_0 \rangle$	P_0	$\frac{4}{7}$	1	1	$7\frac{2}{7}$	$7\frac{2}{7}$	5	45.71%	(4)	$x_1 \leq 0 \rightarrow P_1$ $x_1 \geq 1 \rightarrow P_2$	5 $6\frac{3}{4}$
$\langle P_1, P_2 \rangle$	P_1	0	1	1	5	$6\frac{3}{4}$	5	35.00%	(1)		
$\langle P_2 \rangle$	P_2	1	1	$\frac{1}{4}$	$6\frac{3}{4}$	$6\frac{3}{4}$	5	35.00%	(4)	$x_3 \leq 0 \rightarrow P_3$ $x_3 \geq 1 \rightarrow P_4$	6 n.a.
$\langle P_3, P_4 \rangle$	P_3	1	1	0	6	6	6	0%	(1)		
$\langle P_4 \rangle$	P_4	1	—	1	—				(3)		
$\langle \rangle$											
Opt. sol.		1	1	0	6						

The procedure stops, because the candidate list is empty, with an optimal solution $x_1 = x_2 = 1$ and $x_3 = 0$ and an optimal objective function value of 6. The student buys the necklace and the ring. Also note that the procedure may be terminated once an optimality gap of $\Delta = 0$ has been determined. [After evaluation of P_3]

Question 89

a) 1. Determination of a lower bound \underline{z}

- Determine the value per cost.

Employer	Salary	Hours	Value per Cost
Chair	60	8	7.50
Sieh&Menz	80	8	10
McConsult	90	6	15

- Select the jobs in order of decreasing value per cost while staying within the available hours per week.

Employer	Percentage	cum. Salary	cum. Hours
McConsult	100%	90	6
Sieh&Menz	100%	170	14
Chair	0%	170	14

- We get a feasible solution $x_2 = x_3 = 1$ and $x_1 = 0$ with $\underline{z} = 170$

2. Determination of an upper bound \bar{z}

- Select (part of) the jobs in order of decreasing value per cost while using all available hours per week.

Employer	Percentage	cum. Salary	cum. Hours
McConsult	100%	90	6
Sieh&Menz	100%	170	14
Chair	75%	215	20

- We get an infeasible solution $x_2 = x_3 = 1, x_1 = \frac{3}{4}$ with $\bar{z} = 215$

b)

K	P_{i^*}	x_1	x_2	x_3	z	\bar{z}	\underline{z}	Δ	*)	Branch	UB
$\langle P_0 \rangle$	P_0	$\frac{3}{4}$	1	1	215	215	170	26.47%	(4)	$x_1 \leq 0 \rightarrow P_1$ $x_1 \geq 1 \rightarrow P_2$	170 210
$\langle P_2, P_1 \rangle$	P_2	1	$\frac{3}{4}$	1	210	210	170	23.53%	(4)	$x_2 \leq 0 \rightarrow P_3$ $x_2 \geq 1 \rightarrow P_4$	150 200
$\langle P_4, P_2, P_3 \rangle$	P_4	1	1	$\frac{2}{3}$	200	200	170	17.65%	(4)	$x_3 \leq 0 \rightarrow P_5$ $x_3 \geq 1 \rightarrow P_6$	140 <i>n.feas.</i>
$\langle P_1, P_3, P_5, P_6 \rangle$	P_1	0	1	1	170	170	170	0%	(1)		
$\langle P_3, P_5, P_6 \rangle$	P_3	1	0	1	150				(1)		
$\langle P_5, P_6 \rangle$	P_5	1	1	0	140				(1)		
$\langle P_6 \rangle$	P_6	-	-	-	n. feas.				(3)		
$\langle \rangle$											
Opt. sol.		0	1	1	170						

The procedure stops, because the candidate list is empty, with an optimal solution $x_2 = x_3 = 1$ and $x_1 = 0$ and an optimal objective function value of 170. The student will take the jobs at Sieh&Menz and McConsult. Also note that the procedure may be terminated once an optimality gap of $\Delta = 0$ has been determined. [After evaluation of P_1]

Question 90

a) 1. Determination of a lower bound \underline{z}

- Determine the value per cost of every item.

Item	Value	Weight	Value per Cost
1	8	1	8
2	8	2	4
3	6	2	3
4	10	4	2.5
5	12	6	2
6	12	10	1.2

- Select the items in order of decreasing value per cost while staying within the capacity restriction.

Item	fits?	cum. Value	cum. Weight
1		8	1
2	yes	16	3
3	yes	22	5
4	yes	32	9
5	no	32	9
6	no	32	9

- We get a feasible solution $x_1 = x_2 = x_3 = x_4 = 1$ and $x_5 = x_6 = 0$ with $\underline{z} = 32$

2. Determination of an upper bound \bar{z}

- Select (part of) the items in order of decreasing value per cost while using all of the available capacity.

Item	Percentage	cum. Value	cum. Weight
1	100%	8	1
2	100%	16	3
3	100%	22	5
4	100%	32	9
5	50%	38	12
6	0%	38	12

- We get an infeasible solution $x_1 = x_2 = x_3 = x_4 = 1$, $x_5 = \frac{1}{2}$ and $x_6 = 0$ with $\bar{z} = 38$

b)

K	P_{i^*}	x_1	x_2	x_3	x_4	x_5	x_6	z	\bar{z}	\underline{z}	Δ	*)	Branch	UB
$\langle P_0 \rangle$	P_0	1	1	1	1	$\frac{1}{2}$	0	38	38	32	18.75%	(4)	$x_5 \leq 0 \rightarrow P_1$ $x_5 \geq 1 \rightarrow P_2$	35.6 36.5
$\langle P_1, P_2 \rangle$	P_1	1	1	1	1	0	$\frac{3}{10}$	35.6	36.5	32	14.06%	(4)	$x_6 \leq 0 \rightarrow P_3$ $x_6 \geq 1 \rightarrow P_4$	32 24
$\langle P_3, P_4, P_2 \rangle$	P_3	1	1	1	1	0	0	32				(1)		
$\langle P_4, P_2 \rangle$	P_4	1	$\frac{1}{2}$	0	0	0	1	24				(2)		
$\langle P_2 \rangle$	P_2	1	1	1	$\frac{1}{4}$	1	0	36.5				(4)	$x_4 \leq 0 \rightarrow P_5$ $x_4 \geq 1 \rightarrow P_6$	35.2 34
$\langle P_5, P_6 \rangle$	P_5	1	1	1	0	1	$\frac{1}{10}$	35.2	35.2	32	10.00%	(4)	$x_6 \leq 0 \rightarrow P_7$ $x_6 \geq 1 \rightarrow P_8$	34 n. feas.
$\langle P_7, P_8, P_6 \rangle$	P_7	1	1	1	0	1	0	34	34	34	0.0%	(1)		
$\langle P_8, P_6 \rangle$	P_8	—	—	—	0	1	1	—				(3)		
$\langle P_6 \rangle$	P_6	1	$\frac{1}{2}$	0	1	1	0	34				(2)		
$\langle \rangle$														
Opt. sol.		1	1	1	0	1	0	34						

The procedure stops, because the candidate list is empty, with an optimal solution $x_1 = x_2 = x_3 = x_5 = 1$ and $x_4 = x_6 = 0$ and an optimal objective function value of 34. Also note that the procedure may be terminated once an optimality gap of $\Delta = 0$ has been determined. [After evaluation of P_7]

Question 91

a) 1. Determination of a lower bound \underline{z}

- Determine the value per cost of every order.

Order	Return	Weight	Value per Cost
1	9,000	5	1,800
2	4,000	3	$1,333\frac{1}{3}$
3	3,000	2	1,500
4	11,000	6	$1,833\frac{1}{3}$
5	3,500	2	1,750
6	12,000	7	$1,714\frac{2}{7}$

- Select the orders in order of decreasing value per cost while staying within the capacity restriction.

Order	fits?	cum. Return	cum. Weight
4	yes	11,000	6
1	no	11,000	6
5	yes	14,500	8
6	no	14,500	8
3	yes	17,500	10
2	no	17,500	10

- We get a feasible solution $x_3 = x_4 = x_5 = 1$ and $x_1 = x_2 = x_6 = 0$ with $\underline{z} = 17,500$

2. Determination of an upper bound \bar{z}

- Select (part of) the orders in order of decreasing value per cost while using all of the available capacity.

Order	Percentage	cum. Value	cum. Weight
4	100%	11,000	6
1	80%	18,200	10
5	0%	18,200	10
6	0%	18,200	10
3	0%	18,200	10
2	0%	18,200	10

- We get an infeasible solution $x_4 = 1$, $x_1 = \frac{4}{5}$ and $x_2 = x_3 = x_5 = x_6 = 0$ with $\bar{z} = 18,200$

b)

K	P_{i^*}	x_1	x_2	x_3	x_4	x_5	x_6	z	\bar{z}	\underline{z}	Δ	*)	Branch	UB
$\langle P_0 \rangle$	P_0	$\frac{4}{5}$	0	0	1	0	0	18,200	18,200	17,500	4.00%	(4)	$x_1 \leq 0 \rightarrow P_1$ $x_1 \geq 1 \rightarrow P_2$	17,928 $\frac{4}{7}$ 18,166 $\frac{2}{3}$
$\langle P_2, P_1 \rangle$	P_2	1	0	0	$\frac{5}{6}$	0	0	18,166 $\frac{2}{3}$	18,166 $\frac{2}{3}$	17,500	3.81%	(4)	$x_4 \leq 0 \rightarrow P_3$ $x_4 \geq 1 \rightarrow P_4$	17,642 $\frac{6}{7}$ n. feas.
$\langle P_1, P_3, P_4 \rangle$	P_1	0	0	0	1	1	$\frac{2}{7}$	17,928 $\frac{4}{7}$	17,928 $\frac{4}{7}$	17,500	2.45%	(4)	$x_6 \leq 0 \rightarrow P_5$ $x_6 \geq 1 \rightarrow P_6$	17,500 17,500
$\langle P_3, P_5, P_6, P_4 \rangle$	P_3	1	0	0	0	1	$\frac{3}{7}$	17,642 $\frac{6}{7}$	17,642 $\frac{6}{7}$	17,500	0.82%	(4)	$x_6 \leq 0 \rightarrow P_7$ $x_6 \geq 1 \rightarrow P_8$	16,833 $\frac{1}{3}$ n. feas.
$\langle P_5, P_6, P_7, P_4, P_8 \rangle$	P_5	0	0	1	1	1	0	17,500	17,500	17,500	0%	(1)		
$\langle P_6, P_7, P_4, P_8 \rangle$	P_6	0	0	0	$\frac{1}{2}$	0	1	17,500				(2)		
$\langle P_7, P_4, P_8 \rangle$	P_7	1	$\frac{1}{3}$	1	0	1	0	16,833 $\frac{1}{3}$				(2)		
$\langle P_4, P_8 \rangle$	P_4	1	-	-	1	-	-	-				(3)		
$\langle P_8 \rangle$	P_8	1	-	-	0	-	1	-				(3)		
$\langle \rangle$														
Opt. sol.		0	0	1	1	1	0	17,500						

The procedure stops, because the candidate list is empty, with an optimal solution $x_3 = x_4 = x_5 = 1$ and $x_1 = x_2 = x_6 = 0$ and an optimal objective function value of 17,500. Also note that the procedure may be terminated once an optimality gap of $\Delta = 0$ has been determined. [After evaluation of P_5]

Question 92

a) 1. Determination of an upper bound \bar{z}

- Determine the value per cost.

Alternative	Costs	Decrease	Value per Cost
A	7	5	1.40
B	4	3	1.33
C	6	4	1.5

- Select alternatives in order of decreasing value per cost while achieving the targeted decrease.

Alternative	Percentage	cum. Costs	cum. Decrease
B	100%	4	3
A	100%	11	8
C	0%	11	8

- We get a feasible solution $x_1 = x_2 = 1$ and $x_3 = 0$ with $\bar{z} = 11$

2. Determination of a lower bound \underline{z}

- Select (part of) alternatives in order of decreasing value per cost while reaching exactly the targeted decrease.

Alternative	Percentage	cum. Kosten	cum. Verringerung
B	100%	4	3
A	80%	9.6	7
C	0%	9.6	7

- We get an infeasible solution $x_1 = \frac{4}{5}$, $x_2 = 1$ and $x_3 = 0$ with $\underline{z} = 9.6$

b)

K	P_{i^*}	x_1	x_2	x_3	z	\bar{z}	\underline{z}	Δ	*	Branch	LB
$\langle P_0 \rangle$	P_0	$\frac{4}{5}$	1	0	9.6	11	9.6	12.73%	(4)	$x_1 \leq 0 \rightarrow P_1$	10
										$x_1 \geq 1 \rightarrow P_2$	9.67
$\langle P_2, P_1 \rangle$	P_2	1	$\frac{2}{3}$	0	9.67	11	9.67	12.09%	(4)	$x_2 \leq 0 \rightarrow P_3$	10
										$x_2 \geq 1 \rightarrow P_4$	11
$\langle P_1, P_3, P_4 \rangle$	P_1	0	1	1	10	10	10	0%	(1)		

The procedure stops, because $\Delta = \frac{\bar{z}-\underline{z}}{\bar{z}} = \frac{10-9.6}{10} = 0\%$ is smaller than 4%. An optimal solution $x_2 = x_3 = 1$ and $x_1 = 0$ is found with the optimal objective function value of 10. OZON AG will therefore choose alternatives B and C.

Note: We used \bar{z} in the denominator in the gap calculation since this is a minimization problem and the incumbent is saved in the upper bound in the minimization problem.

Question 93

a) 1. Determination of a lower bound \underline{z}

- A decision about building or not building the warehouses is not made, meaning $y_1, y_2 \in \{0,1\}$
- Each store gets its complete demand from a warehouse which could still be built with the lowest transportation cost per unit.
- The total costs are made up of the fixed costs for building the warehouses and the transportation costs. As there is no decision made about any warehouse being built or not, we do not have any fixed costs and get an infeasible solution of $\underline{z} = 11 \cdot 7 + 14 \cdot 5 + 9 \cdot 6 = 201$

2. Determination of an upper bound \bar{z}

- A decision is now made to build all of the warehouses meaning $y_1 = y_2 = 1$ and each store gets its complete demand from the warehouse with the lowest transportation costs again.
- The total costs are determined as they were for the determination of a lower bound and we get a feasible solution with $\bar{z} = 85 + 70 + 11 \cdot 7 + 14 \cdot 5 + 9 \cdot 6 = 356$

b)

K	P_{i^*}	y_1	y_2	z	\bar{z}	\underline{z}	Δ	*	Branch	LB
$\langle P_0 \rangle$	P_0	{0,1}	{0,1}	201	356	201	43.54%	(4)	$y_1 \leq 0 \rightarrow P_1$ $y_1 \geq 1 \rightarrow P_2$	275 286
$\langle P_2, P_1 \rangle$	P_2	1	{0,1}	286	356	275	22.75%	(4)	$y_2 \leq 0 \rightarrow P_3$ $y_2 \geq 1 \rightarrow P_4$	297 356
$\langle P_4, P_3, P_1 \rangle$	P_4	1	1	356				(1)		
$\langle P_3, P_1 \rangle$	P_3	1	0	297	297	275	7.41%	(1)		
$\langle P_1 \rangle$	P_1	0	{0,1}	275				(4)	$y_2 \leq 0 \rightarrow P_5$ $y_2 \geq 1 \rightarrow P_6$	n.feas. 345
$\langle P_6, P_5 \rangle$	P_6	0	1	345	297	297	0%	(1)		
$\langle P_5 \rangle$	P_5	0	0	—				(3)		
$\langle \rangle$										
Opt. sol.		1	0	297						

The procedure stops, because the candidate list K is empty, with an optimal solution $y_1 = 1$ and $y_2 = 0$ and an optimal objective function value of 297. Only the first furniture warehouse is built.

Question 94

- a) 1. Determination of a lower bound \underline{z}
- A decision about building or not building the warehouses is not made, meaning $y_1, y_2, y_3 \in \{0,1\}$
 - Each store gets its complete demand from a warehouse which could still be built with the lowest transportation cost per unit.
 - The total costs are made up of the fixed costs for building the warehouses and the transportation costs. As there is no decision made about any warehouse being built or not, we do not have any fixed costs and get an infeasible solution of $\underline{z} = 10 \cdot 9 + 17 \cdot 6 + 21 \cdot 6 + \dots = 715$
2. Determination of an upper bound \bar{z}
- A decision is now made to build all of the warehouses meaning $y_1 = y_2 = y_3 = 1$ and each store gets its complete demand from the warehouse with the lowest transportation costs again.
 - The total costs are determined as they were for the determination of a lower bound and we get a feasible solution with $\bar{z} = 97 + 74 + 83 + 10 \cdot 9 + 17 \cdot 6 + 21 \cdot 6 + \dots = 969$

b)

K	P_{i^*}	y_1	y_2	y_3	z	\bar{z}	\underline{z}	Δ	*)	Branch	LB
$\langle P_0 \rangle$	P_0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	715	969	715	26.21%	(4)	$y_1 \leq 0 \rightarrow P_1$	753
										$y_1 \geq 1 \rightarrow P_2$	812
$\langle P_1, P_2 \rangle$	P_1	0	$\{0,1\}$	$\{0,1\}$	753	969	753	22.29 %	(4)	$y_2 \leq 0 \rightarrow P_3$	996
										$y_2 \geq 1 \rightarrow P_4$	827

The procedure is stopped before the candidate list K is empty, because the the maximum deviation of the objective function value of the best known feasible solution from the theoretically best achievable objective function value is with $\Delta = \frac{\bar{z}-\underline{z}}{\bar{z}} = \frac{969-753}{969} = 22.21\%$ not greater than 25%.

Question 95

- a) Maximization problem, as the objective function value decreases with increasing depth of the branch-and-bound-Tree.
- b) MUB with biggest number tie-breaker, because the sub-problem with the higher objective function value is selected first.
- c) Variable x_2 is branched as it is the only variable in the optimal solution of the sub-problem P_3 with a fractional (non-integer) value.
- d) (1) P_1 is not branched, because the solution is feasible for the initial problem (case 1). The lower bound is updated to the new value $\underline{z} = 50$.
 (2) P_4 and P_6 are not branched, because the solution of the relaxed problem is infeasible (case 3).
 (3) P_5 is not branched, because the solution is feasible for the initial problem (case 1). The lower bound is not updated as $z = 40$ is smaller than the lower bound $\underline{z} = 50$.
- e) Optimal solution $x_1 = x_2 = 1$ and $x_3 = 0$ with objective function value 50.
- f) The procedure would stop with the in e) stated solution after the computation of sub-problem P_1 .

It is then certain that the maximum deviation of the best known feasible solution from a possibly unknown optimal solution is $\Delta = \frac{\bar{z}-\underline{z}}{\underline{z}} = \frac{50-50}{50} = 0\%$.

- g) Maximize

$$30 \cdot x_1 + 20 \cdot x_2 + 40 \cdot x_3 = z$$

subject to

$$2 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 \leq 4$$

$$x_1, x_2, x_3 \in \{0,1\}$$

Question 96

- a) Maximization problem, as the objective function value decreases with increasing depth of the branch-and-bound-Tree.
- b) LIFO with biggest number tie-breaker, because the last solved sub-problem is always branched no matter the objective function value.
- c) Variable x_3 is branched as it is the only variable in the optimal solution of the sub-problem P_6 with a fractional (non-integer) value.
- d) case (1): $P_7, P_9, P_{10}, P_{12}, P_{14}$
 case (2): P_{15}
 case (3): P_{11}, P_{13}
- e) Optimal solution $x_1 = x_2 = x_3 = x_5 = 1$ and $x_4 = x_6 = 0$ with objective function value 34.
- f) The procedure would stop after it has been determined that P_7 is integer feasible (in iteration 4). It is then certain that the maximum deviation of the best known feasible solution from a possibly unknown optimal solution is $\Delta = \frac{\bar{z}-\underline{z}}{\underline{z}} = \frac{35.6-30}{30} = 18.67\%$. The progress on the optimality gap looks as follows: Assume an incumbent solution with objective function value of 20 is known. At the beginning of the algorithm, the upper bound $\bar{z} = 38$ is given by the objective function value of the root node P_1 . After branching on problem P_1 , the upper bound \bar{z} can be reduced to 36.5. After branching on P_3 , the upper bound \bar{z} can be further reduced to 35.6 [determined by problem P_2]. In iteration 4, it is determined that the solution to P_7 is integer feasible, improving the lower bound to $\underline{z} = 30$. At this stage of the algorithm, the upper bound is still 35.6 [determined by the objective function values of the open problems P_2, P_4, P_6 and P_7]. With $\bar{z} = 35.6$ and $\underline{z} = 30$ it follows that $\Delta = 18.67\% \leq 20\%$.

Iteration	\bar{z}	\underline{z}	Δ
1	38	20	90.00%
2	36.5	20	82.5%
3	35.6	20	78.00%
4	35.6	30	18.67%

4 Graph Theory and Network Flow Models

4.1 Shortest path problems: Dijkstra's algorithm

Question 97

- Any graph (directed or undirected)
- The graph has no negative arc weights.

Question 98

There are n iterations needed, because Dijkstra's algorithm determines the shortest path from the starting node to one other node of the graph in each iteration.

Question 99

Start

i	1	2	3	4	5	6
$d[i]$	0	∞	∞	∞	∞	∞
$p[i]$						
	$M=\{1\}$					

Iteration 1 ($h = 1$)

i	1	2	3	4	5	6
$d[i]$	0	5	4	∞	∞	∞
$p[i]$		1	1			
	$M=\{2, 3\}$					

Iteration 2 ($h = 3$)

i	1	2	3	4	5	6
$d[i]$	0	5	4	∞	7	∞
$p[i]$		1	1		3	
	$M=\{2,5\}$					

Iteration 3 ($h = 2$)

i	1	2	3	4	5	6
$d[i]$	0	5	4	11	7	∞
$p[i]$		1	1	2	3	
	$M=\{4,5\}$					

Iteration 4 ($h = 5$)

i	1	2	3	4	5	6
$d[i]$	0	5	4	9	7	11
$p[i]$		1	1	5	3	5
	$M=\{4,6\}$					

Iteration 5 ($h = 4$)

i	1	2	3	4	5	6
$d[i]$	0	5	4	9	7	10
$p[i]$		1	1	5	3	4
	$M=\{6\}$					

Iteration 6 ($h = 6$)

i	1	2	3	4	5	6
$d[i]$	0	5	4	9	7	10
$p[i]$		1	1	5	3	4
	$M=\{\}$					

Iteration 7 \rightarrow The algorithm stops, because set M is empty. The shortest path from node 1 to node 6 has a length of 10 and follows the path:

$1 \rightarrow 3 \rightarrow 5 \rightarrow 4 \rightarrow 6$

Question 100

Start

i	1	2	3	4
$d[i]$	0	∞	∞	∞
$p[i]$				
	$M=\{1\}$			

Iteration 1 ($h = 1$)

i	1	2	3	4
$d[i]$	0	2	5	∞
$p[i]$		1	1	
	$M=\{2, 3\}$			

Iteration 2 ($h = 2$)

i	1	2	3	4
$d[i]$	0	2	3	7
$p[i]$		1	2	2
	$M=\{3,4\}$			

Iteration 3 ($h = 3$)

i	1	2	3	4
$d[i]$	0	2	3	6
$p[i]$		1	2	3
	$M=\{4\}$			

Iteration 4 ($h = 4$)

i	1	2	3	4
$d[i]$	0	2	3	6
$p[i]$		1	2	3
	$M = \{\}$			

Iteration 5 \rightarrow The algorithm stops, because set M is empty. The shortest path from node 1 to node 4 has a length of 6 and follows the path:

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$

Question 101

a)

Start

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	∞													
$p[i]$															
	$M = \{1\}$														

Iteration 1 ($h = 1$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	∞	190	52	∞									
$p[i]$		1		1	1										
	$M = \{2,4,5\}$														

Iteration 2 ($h = 5$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	∞	190	52	128	171	∞							
$p[i]$		1		1	1	5	5								
	$M = \{2,4,6,7\}$														

Iteration 3 ($h = 6$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	∞	190	52	128	144	∞	224	∞	∞	∞	∞	∞	∞
$p[i]$		1		1	1	5	6		6						
	$M = \{2,4,7,9\}$														

Iteration 4 ($h = 2$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	193	190	52	128	144	∞	224	∞	∞	∞	∞	∞	∞
$p[i]$		1	2	1	1	5	6		6						
	$M = \{3,4,7,9\}$														

Iteration 5 ($h = 7$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	193	190	52	128	144	∞	224	219	∞	∞	∞	∞	∞
$p[i]$		1	2	1	1	5	6		6	7					
$M = \{3,4,9,10\}$															

Iteration 6 ($h = 4$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	193	190	52	128	144	∞	224	219	∞	∞	∞	∞	∞
$p[i]$		1	2	1	1	5	6		6	7					
$M = \{3,9,10\}$															

Iteration 7 ($h = 3$)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d[i]$	0	138	193	190	52	128	144	341	224	219	∞	∞	∞	∞	∞
$p[i]$		1	2	1	1	5	6	3	6	7					
$M = \{8,9,10\}$															

The algorithm can be stopped, because node 3 has been chosen in iteration 7 which means the shortest path from node 1 to node 3 is established. The shortest path from 1 to 3 goes through node 2 and has a length of 193.

- b) The algorithm could have been stopped after iteration 5, because node 7 was chosen in that iteration.

Question 102

i	1	2	3	4	5	6	7	8
$d[i]$	0	∞						
$p[i]$								
$M = \{1\}$								

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	∞	∞	∞	∞	∞	∞
$p[i]$		1						
$M = \{2\}$								

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	∞	∞	5	∞	∞
$p[i]$		1	2			2		
$M = \{3,6\}$								

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	∞	9	5	∞	∞
$p[i]$		1	2		3	2		
	$M=\{5,6\}$							

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	7	8.5	5	9	∞
$p[i]$		1	2	6	6	2	6	
	$M=\{4,5,7\}$							

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	7	7.5	5	9	∞
$p[i]$		1	2	6	4	2	6	
	$MK=\{5,7\}$							

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	7	7.5	5	8.5	10
$p[i]$		1	2	6	4	2	5	5
	$M=\{7,8\}$							

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	7	7.5	5	8.5	9.5
$p[i]$		1	2	6	4	2	5	7
	$M=\{8\}$							

i	1	2	3	4	5	6	7	8
$d[i]$	0	2	4.5	7	7.5	5	8.5	9.5
$p[i]$		1	2	6	4	2	5	7
	$M=\emptyset$							

The shortest path from Hauptbahnhof to Dt.Museum has a length of 9.5 and follows the path:

Hauptbahnhof \rightarrow Stachus \rightarrow Sendlinger Tor \rightarrow Marienplatz \rightarrow Isartor \rightarrow Gärtnerplatz \rightarrow Dt. Museum

Question 103

a) Iteration 2 ($h = 4$)

i	1	2	3	4	5
$d[i]$	0	2	3	1	3
$p[i]$		4	1	1	4
$M = \{2,3,5\}$					

b) Iteration 3 ($h = 2$)

i	1	2	3	4	5
$d[i]$	0	2	3	1	3
$p[i]$		4	1	1	4
$M = \{3,5\}$					

The shortest paths from node 1 to node 2 and from node 1 to node 4 are known. The shortest paths from node 1 to node 3 and from node 1 to node 5 are still unknown as they are still in the set $M = \{3,5\}$.

4.2 Shortest path problems: Floyd-Warshall algorithm

Question 104

Note: The entries of the matrices are: $d_{ij}; p_{ij}$

Initialization:

	1	2	3	4	5	6
1	<u>0;1</u>	2;1	6;1	$\infty;0$	$\infty;0$	$\infty;0$
2	$\infty;0$	<u>0;2</u>	$\infty;0$	8;2	2;2	$\infty;0$
3	$\infty;0$	$\infty;0$	<u>0;3</u>	3;3	3;3	$\infty;0$
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	$\infty;0$	3;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>	2;5
6	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;6</u>

Result after iteration $v = 2$:

	1	2	3	4	5	6
1	<u>0;1</u>	2;1	6;1	10;2	4;2	$\infty;0$
2	$\infty;0$	<u>0;2</u>	$\infty;0$	8;2	2;2	$\infty;0$
3	$\infty;0$	$\infty;0$	<u>0;3</u>	3;3	3;3	$\infty;0$
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	$\infty;0$	3;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>	2;5
6	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;6</u>

Result after Iteration $v = 3$:

	1	2	3	4	5	6
1	<u>0;1</u>	2;1	6;1	<u>9;3</u>	4;2	$\infty;0$
2	$\infty;0$	<u>0;2</u>	$\infty;0$	8;2	2;2	$\infty;0$
3	$\infty;0$	$\infty;0$	<u>0;3</u>	3;3	3;3	$\infty;0$
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	$\infty;0$	3;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>	2;5
6	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;6</u>

Result after iteration $v = 4$:

	1	2	3	4	5	6
1	<u>0;1</u>	2;1	6;1	9;3	4;2	<u>12;4</u>
2	$\infty;0$	<u>0;2</u>	$\infty;0$	8;2	2;2	<u>11;4</u>
3	$\infty;0$	$\infty;0$	<u>0;3</u>	3;3	3;3	<u>6;4</u>
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	$\infty;0$	3;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>	2;5
6	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;6</u>

Result after iteration $v = 5$:

	1	2	3	4	5	6
1	<u>0;1</u>	2;1	6;1	9;3	4;2	<u>6;5</u>
2	$\infty;0$	<u>0;2</u>	$\infty;0$	8;2	2;2	<u>4;5</u>
3	$\infty;0$	$\infty;0$	<u>0;3</u>	3;3	3;3	<u>5;5</u>
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	$\infty;0$	3;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>	2;5
6	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;6</u>

The shortest distance from node 1 to node 6 has a length of $d_{16} = 6$.

The path from node 1 to node 6 with the shortest distance is recursively determined through $p_{16} = 5$, $p_{15} = 2$ and $p_{12} = 1$ to $((1,2),(2,5),(5,6))$.

Question 105

Note: The entries of the matrices are $d_{ij}; p_{ij}$

Initialization:

	1	2	3	4	5
1	<u>0;1</u>	<u>2;1</u>	$\infty;0$	<u>6;1</u>	$\infty;0$
2	$\infty;0$	<u>0;2</u>	<u>7;2</u>	$\infty;0$	$\infty;0$
3	$\infty;0$	$\infty;0$	<u>0;3</u>	$\infty;0$	<u>5;3</u>
4	$\infty;0$	$\infty;0$	<u>2;4</u>	<u>0;4</u>	$\infty;0$
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

Result after iteration $v = 2$:

	1	2	3	4	5
1	<u>0;1</u>	2;1	9;2	6;1	$\infty;0$
2	$\infty;0$	<u>0;2</u>	7;2	$\infty;0$	$\infty;0$
3	$\infty;0$	$\infty;0$	<u>0;3</u>	$\infty;0$	5;3
4	$\infty;0$	$\infty;0$	2;4	<u>0;4</u>	$\infty;0$
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

Result after iteration $v = 3$:

	1	2	3	4	5
1	<u>0;1</u>	2;1	9;2	6;1	14;3
2	$\infty;0$	<u>0;2</u>	7;2	$\infty;0$	12;3
3	$\infty;0$	$\infty;0$	<u>0;3</u>	$\infty;0$	5;3
4	$\infty;0$	$\infty;0$	2;4	<u>0;4</u>	7;3
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

Result after iteration $v = 4$:

	1	2	3	4	5
1	<u>0;1</u>	2;1	8;4	6;1	13;3
2	$\infty;0$	<u>0;2</u>	7;2	$\infty;0$	12;3
3	$\infty;0$	$\infty;0$	<u>0;3</u>	$\infty;0$	5;3
4	$\infty;0$	$\infty;0$	2;4	<u>0;4</u>	7;3
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

The shortest path from node 1 to node 5 has a length of $d_{15} = 13$.

The path from node 1 to node 5 with the shortest distance is recursively determined through $p_{15} = 3$, $p_{13} = 4$ and $p_{14} = 1$ to $((1,4),(4,3),(3,5))$.

Question 106

Note: The entries of the matrices are $d_{ij}; p_{ij}$

Initialization:

	1	2	3	4	5
1	<u>0;1</u>	1;1	3;1	6;1	$\infty;0$
2	$\infty;0$	<u>0;2</u>	1;2	$\infty;0$	8;2
3	$\infty;0$	$\infty;0$	<u>0;3</u>	2;3	4;3
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	1;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

Result after iteration $v = 2$:

	1	2	3	4	5
1	<u>0;1</u>	1;1	2;2	6;1	9;2
2	$\infty;0$	<u>0;2</u>	1;2	$\infty;0$	8;2
3	$\infty;0$	$\infty;0$	<u>0;3</u>	2;3	4;3
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	1;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

Result after iteration $v = 3$:

	1	2	3	4	5
1	<u>0;1</u>	1;1	2;2	4;3	6;3
2	$\infty;0$	<u>0;2</u>	1;2	3;3	5;3
3	$\infty;0$	$\infty;0$	<u>0;3</u>	2;3	4;3
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	1;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

Result after iteration $v = 4$:

	1	2	3	4	5
1	<u>0;1</u>	1;1	2;2	4;3	5;4
2	$\infty;0$	<u>0;2</u>	1;2	3;3	4;4
3	$\infty;0$	$\infty;0$	<u>0;3</u>	2;3	3;4
4	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;4</u>	1;4
5	$\infty;0$	$\infty;0$	$\infty;0$	$\infty;0$	<u>0;5</u>

The shortest path from node 1 to node 5 has a length of $d_{15} = 5$.

The path from node 1 to node 5 with the shortest distance is recursively determined through $p_{15} = 4$, $p_{14} = 3$, $p_{13} = 2$ and $p_{12} = 1$ to $((1,2),(2,3),(3,4),(4,5))$.

4.3 Minimum spanning tree problems: Kruskal's algorithm

Question 107

$$L = \langle [1,3],[2,4],[1,2],[2,3],[3,4],[4,5],[3,5] \rangle$$

Iteration	[i,j]	A'	$\sum_{[i,j] \in A'} c_{i,j}$
0	---	---	0
1	[1,3]	{[1,3]}	2
2	[2,4]	{[1,3],[2,4]}	4
3	[1,2]	{[1,3],[2,4],[1,2]}	7
4	[2,3]	{[1,3],[2,4],[1,2]}	7
5	[3,4]	{[1,3],[2,4],[1,2]}	7
6	[4,5]	{[1,3],[2,4],[1,2],[4,5]}	12

The algorithm is stopped, because a minimum spanning tree for a graph with n nodes has exactly $n-1$ edges. The graph has 5 nodes and the algorithm is stopped after the fourth edge is added.

Question 108

$$L = \langle [2,4],[2,3],[3,4],[1,2],[1,4] \rangle$$

Iteration	[i,j]	A'	$\sum_{[i,j] \in A'} c_{i,j}$
0	---	---	0
1	[2,4]	{[2,4]}	2
2	[2,3]	{[2,4],[2,3]}	5
3	[3,4]	{[2,4],[2,3]}	5
4	[1,2]	{[2,4],[2,3],[1,2]}	10

The algorithm is stopped, because a minimum spanning tree for a graph with n nodes has exactly $n-1$ edges. The graph has 4 nodes and the algorithm is stopped after the third edge is added.

Question 109

$$L = \langle [4,6],[6,7],[7,8],[1,2],[4,5],[2,3],[6,8],[2,5],[5,6],[5,7],[3,6] \rangle$$

Iteration	[i,j]	A'	$\sum_{[i,j] \in A'} c_{i,j}$
0	---	---	0
1	[4,6]	{[4,6]}	0.5
2	[6,7]	{[4,6],[6,7]}	1.5
3	[7,8]	{[4,6],[6,7],[7,8]}	2.5
4	[1,2]	{[4,6],[6,7],[7,8],[1,2]}	4.5
5	[4,5]	{[4,6],[6,7],[7,8],[1,2],[4,5]}	6.5
6	[2,3]	{[4,6],[6,7],[7,8],[1,2],[4,5],[2,3]}	9
7	[6,8]	{[4,6],[6,7],[7,8],[1,2],[4,5],[2,3]}	9
8	[2,5]	{[4,6],[6,7],[7,8],[1,2],[4,5],[2,3],[2,5]}	12

Question 110

a)

$$L = \langle [6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[10,11],[11,15],[7,10],[8,12],[5,6], \\ [11,12],[3,4],[6,9],[8,9],[5,7],[1,2],[3,8],[1,4],[13,14],[8,13] \rangle$$

Iteration	[i,j]	A'	$\sum_{[i,j] \in A'} c_{i,j}$
0	–	–	0
1	[6,7]	{[6,7]}	16
2	[9,10]	{[6,7],[9,10]}	45
3	[4,9]	{[6,7],[9,10],[4,9]}	87
4	[12,14]	{[6,7],[9,10],[4,9],[12,14]}	137
5	[1,5]	{[6,7],[9,10],[4,9],[12,14],[1,5]}	189
6	[14,15]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15]}	242
7	[2,3]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3]}	297
8	[9,11]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11]}	354
9	[10,11]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11]}	354
10	[11,15]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15]}	427
11	[7,10]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10]}	502
12	[8,12]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12]}	577
13	[5,6]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6]}	653
14	[11,12]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6]}	653
15	[3,4]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
16	[6,9]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
17	[8,9]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
18	[5,7]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
19	[1,2]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
20	[3,8]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
21	[1,4]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4]}	746
22	[13,14]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4],[13,14]}	972
23	[8,13]	{[6,7],[9,10],[4,9],[12,14],[1,5],[14,15],[2,3],[9,11],[11,15],[7,10],[8,12],[5,6],[3,4],[13,14]}	972

b) A minimum spanning tree in a graph with n nodes has $n - 1$ edges.

4.4 Transportation problem: Northwest corner rule

Question 111

	1	2	3	4	a_i				
1	6				6	0			
2	1				1	0			
3		5	3	2	10	5	2	0	
b_j	7	5	3	2					

1 0 0 0

0

$$z = 2 \cdot 6 + 1 \cdot 1 + 8 \cdot 5 + 15 \cdot 3 + 9 \cdot 2 = 116$$

Question 112

	1	2	3	4	a_i				
1	3	2			5	2	0		
2		4	4		8	4	0		
3			2	5	7	5	0		
b_j	3	6	6	5					

0 4 2 0

0 0

$$z = 4 \cdot 3 + 5 \cdot 2 + 8 \cdot 4 + 10 \cdot 4 + 4 \cdot 2 + 3 \cdot 5 = 117$$

Question 113

	1	2	3	4	5	a_i				
1	30	40	20	10		100	70	30	10	0
2					50	50	0			
3					100	100	0			
b_j	30	40	20	10	150					

0 0 0 0 100

0

$$z = 3 \cdot 30 + 4 \cdot 40 + 2 \cdot 20 + 1 \cdot 10 = 300$$

4.5 Allocation problem: Northwest corner rule

Question 114

a)

	1	2	3	4	5	a_i	
1	1					1	0
2		1				1	0
3			1			1	0
4				1		1	0
5					1	1	0
b_j	1	1	1	1	1		
	0	0	0	0	0		

$$z = 4 \cdot 1 + 3 \cdot 1 + 5 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 = 19$$

b) *Is not discussed in the lecture.*

Question 115

a)

	1	2	3	4	a_i	
1	1	1			2	1 0
2			1	1	2	1 0
b_j	1	1	1	1		
	0	0	0	0		

$$z = 3.6 \cdot 1 + 7.8 \cdot 1 + 3.1 \cdot 1 + 4.9 \cdot 1 = 19.4$$

b) *Is not discussed in the lecture.*

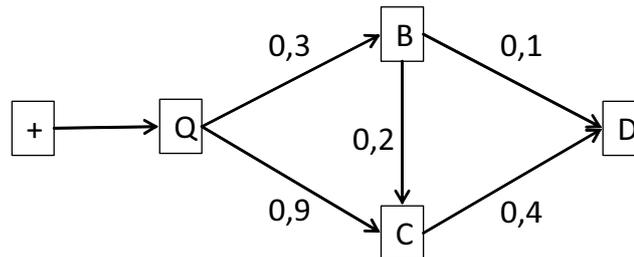
4.6 Maximum flow problems: Ford-Fulkerson algorithm

Question 116

- Directed graph with a source and a sink as well as minimum and/or maximum capacities.
- Feasible start solution must be known.

Question 117

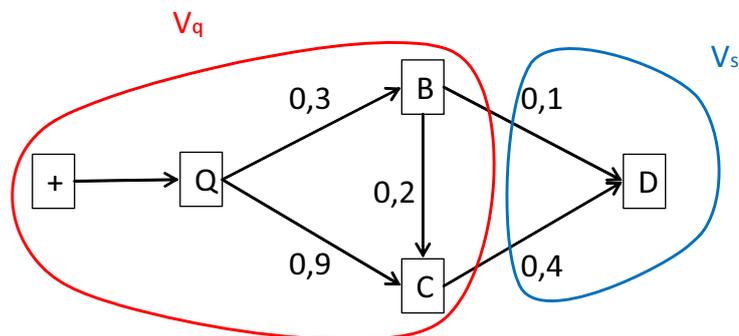
a)



b)

Node	Start solution	Iteration			
		1	2	3	4
Q	(+,0)	(+,∞)	(+,∞)	(+,∞)	(+,∞)
B	(Q ⁺ ,0)	(Q ⁺ ,3)	(Q ⁺ ,2)		(C ⁻ ,2)
C	(Q ⁺ ,0),(B ⁺ ,0)		(B ⁺ ,2)	(Q ⁺ ,9)	(Q ⁺ ,7)
D	(B ⁺ ,0),(C ⁺ ,0)	(B ⁺ ,1)	(C ⁺ ,2)	(C ⁺ ,2)	
Δ	0	1	2	2	0
Σ	0	1	3	5	5

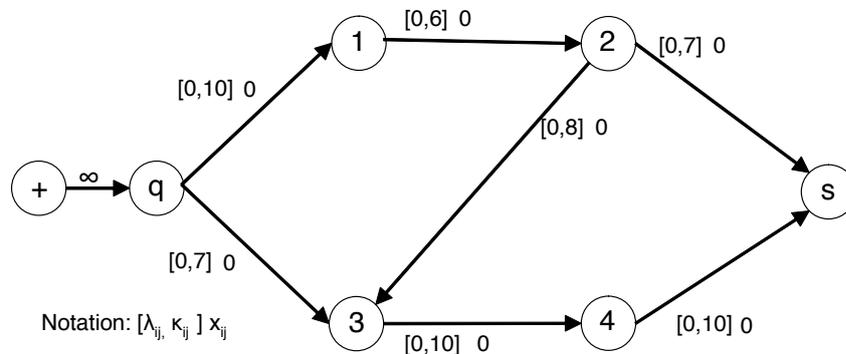
c)



The minimum capacity of the graph is 5.

Question 118

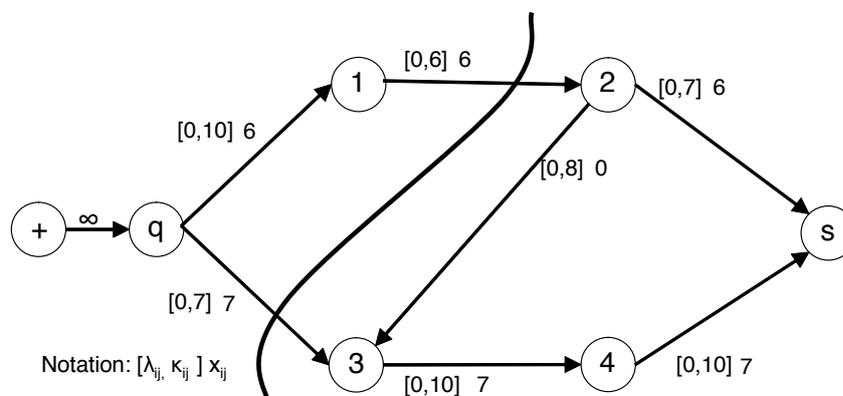
a)



b)

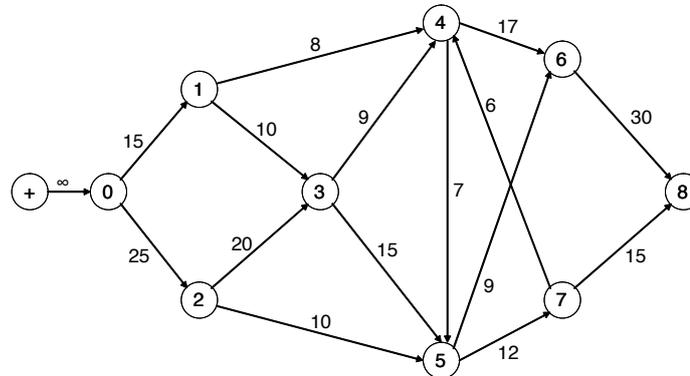
		Iteration			
Node	Start solution	1	2	3	4
q	(+,0)	(+,∞)	(+,∞)	(+,∞)	(+,∞)
1	(q ⁺ ,0)	(q ⁺ ,10)			(q ⁺ ,4)
2	(1 ⁺ ,0)	(1 ⁺ ,6)	(3 ⁻ ,6)		
3	(q ⁺ ,0),(2 ⁺ ,0)	(2 ⁺ ,6)	(q ⁺ ,7)	(q ⁺ ,1)	
4	(3 ⁺ ,0)	(3 ⁺ ,6)		(3 ⁺ ,1)	
s	(2 ⁺ ,0),(4 ⁺ ,0)	(4 ⁺ ,6)	(2 ⁺ ,6)	(4 ⁺ ,1)	
Δ	0	6	6	1	0
Σ	0	6	12	13	0

c) $V_q = \{q,1\}$, $V_s = \{2,3,4,s\}$, capacity $K = 13$

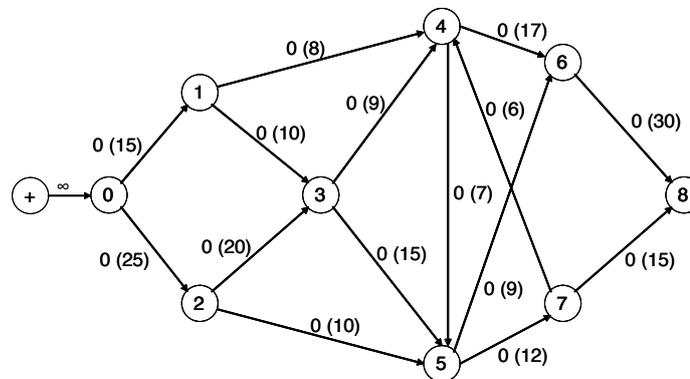


Question 119

a)



b)



c)

Node	Start solution	Iteration						
		1	2	3	4	5	6	7
$q = 0$	$(+,0)$	$(+,∞)$	$(+,∞)$	$(+,∞)$	$(+,∞)$	$(+,∞)$	$(+,∞)$	$(+,∞)$
1	$(0^+,0)$	$(0^+,15)$	$(0^+,6)$				$(3^-,4)$	$(3^-,2)$
2	$(0^+,0)$			$(0^+,25)$	$(0^+,15)$	$(0^+,13)$	$(0^+,4)$	$(0^+,2)$
3	$(1^+,0),(2^+,0)$	$(1^+,10)$			$(2^+,15)$	$(2^+,13)$	$(2^+,4)$	$(2^+,2)$
4	$(1^+,0),(3^+,0),(7^+,0)$	$(3^+,9)$	$(1^+,6)$				$(1^+,2)$	
5	$(2^+,0),(3^+,0),(4^+,0)$			$(2^+,10)$	$(3^+,15)$	$(3^+,13)$		$(3^+,2)$
6	$(4^+,0),(5^+,0)$	$(4^+,9)$	$(4^+,6)$			$(5^+,9)$	$(4^+,2)$	
7	$(5^+,0)$			$(5^+,10)$	$(5^+,2)$			
$s = 8$	$(6^+,0),(7^+,0)$	$(6^+,9)$	$(6^+,6)$	$(7^+,10)$	$(7^+,2)$	$(6^+,9)$	$(6^+,2)$	
Δ	0	9	6	10	2	9	2	
Σ	0	9	15	25	27	36	38	

Question 120

- a) refer to script
- b) refer to graph 1
- c) The maximum flow is reduced if the maximum capacity of one of the arcs $(1,4),(3,4),(5,6),(5,7)$ is reduced and/or the minimum capacity of arc $(4,5)$ increased.

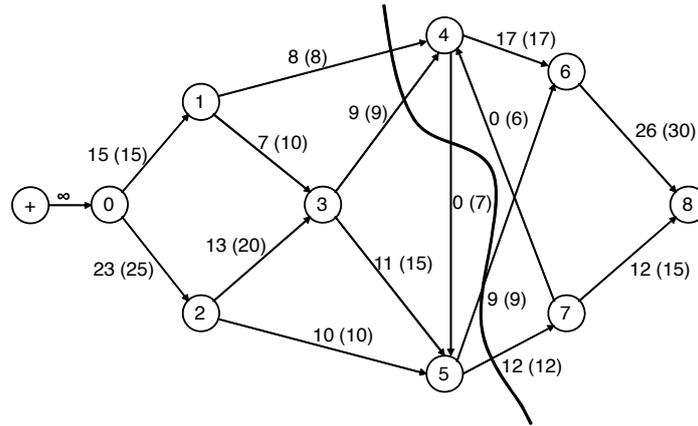
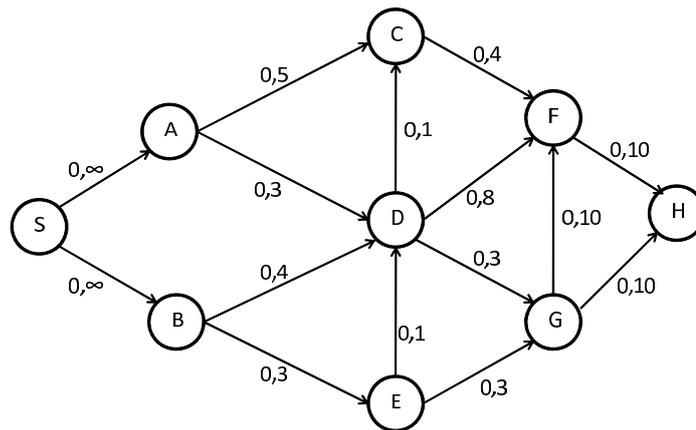


Abbildung 1: graph question b)

Question 121

a)

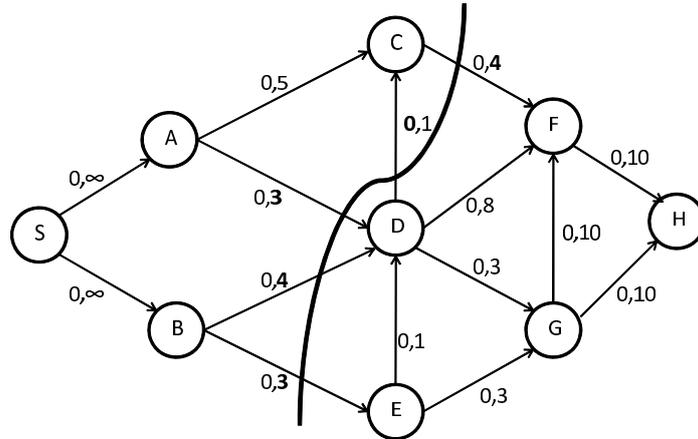


b) refer to (c)

c)

Node	Start solution	Iteration					
		1	2	3	4	5	6
S	(+,0)	(+,∞)	(+,∞)	(+,∞)	(+,∞)	(+,∞)	(+,∞)
A	(S ⁺ ,0)	(S ⁺ ,∞)	(S ⁺ ,∞)				(S ⁺ ,∞)
B	(S ⁺ ,0)			(S ⁺ ,∞)	(S ⁺ ,∞)	(S ⁺ ,∞)	(S ⁺ ,∞)
C	(A ⁺ ,0),(D ⁺ ,0)	(A ⁺ ,5)					(A ⁺ ,1)
D	(A ⁺ ,0),(B ⁺ ,0),(E ⁺ ,0)		(A ⁺ ,3)	(B ⁺ ,4)		(B ⁺ ,1)	
E	(B ⁺ ,0)				(B ⁺ ,3)		
F	(C ⁺ ,0),(D ⁺ ,0),(G ⁺ ,0)	(C ⁺ ,4)	(D ⁺ ,3)		(G ⁺ ,3)	(D ⁺ ,1)	
G	(D ⁺ ,0),(E ⁺ ,0)			(D ⁺ ,3)	(E ⁺ ,3)	(F ⁻ ,1)	
H	(F ⁺ ,0),(G ⁺ ,0)	(F ⁺ ,4)	(F ⁺ ,3)	(G ⁺ ,3)	(F ⁺ ,3)	(G ⁺ ,1)	
Δ	0	4	3	3	3	1	
Σ	0	4	7	10	13	14	

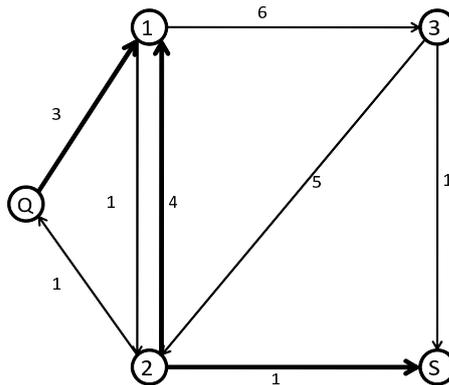
d)



The maximum capacity of the water pipes (B,E) , (B,D) , (A,D) , (C,F) must not be reduced.

Question 122

a)



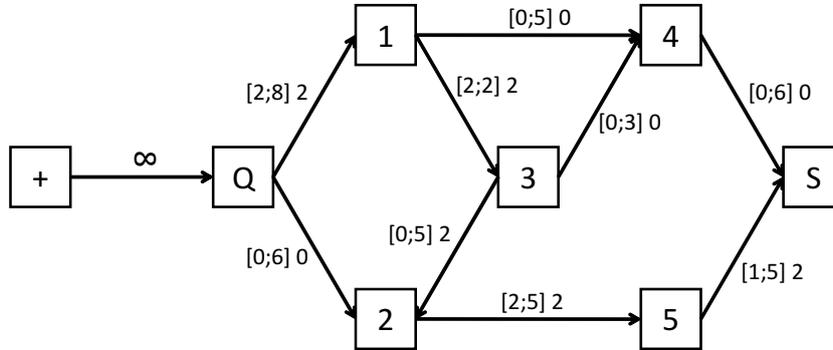
b)

Node	Start solution	Iteration	
		1	2
Q	$(+,2), (2^+,1)$	$(+,∞)$	$(+,∞)$
1	$(Q^+,3), (2^+,4)$	$(Q^+,3)$	$(Q^+,1)$
2	$(1^+,1), (3^+,5)$	$(1^-,2)$	
3	$(1^+,6)$		
S	$(2^+,1), (3^+,1)$	$(2^+,2)$	
Δ	2	2	
Σ	2	4	

c) The maximum capacity of the water pipes $(1,2)$, $(1,3)$ and the minimum capacity of the water pipes $(Q,2)$, $(2,1)$.

Question 123

a)

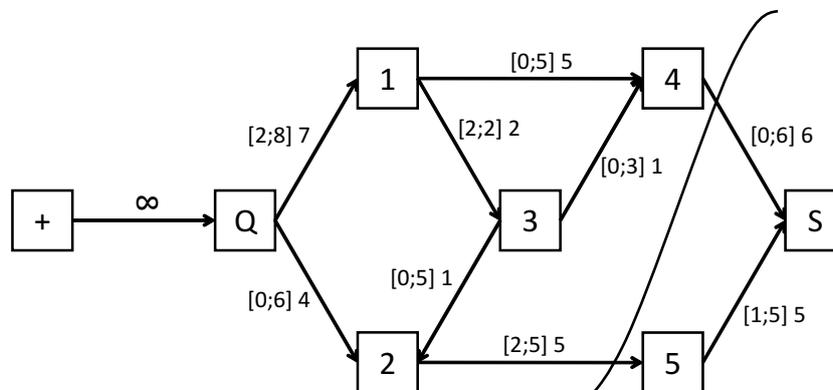


Notation: $[\lambda_i, k_i] x_i$

b)

Node	Start solution	Iteration			
		1	2	3	4
Q	(+,2)	(+,∞)	(+,∞)	(+,∞)	(+,∞)
1	(Q ⁺ ,2)		(Q ⁺ ,6)		(Q ⁺ ,1)
2	(Q ⁺ ,0),(3 ⁺ ,2)	(Q ⁺ ,6)		(Q ⁺ ,3)	(Q ⁺ ,2)
3	(1 ⁺ ,2)			(2 ⁻ ,2)	(2 ⁻ ,1)
4	(1 ⁺ ,0),(3 ⁺ ,0)		(1 ⁺ ,5)	(3 ⁺ ,2)	(3 ⁺ ,1)
5	(2 ⁺ ,2)	(2 ⁺ ,3)			
S	(5 ⁺ ,2),(4 ⁺ ,0)	(5 ⁺ ,3)	(4 ⁺ ,5)	(4 ⁺ ,1)	
Δ	2	3	5	1	0
Σ	2	5	10	11	11

c)



Notation: $[\lambda_i, k_i] x_i$

$$V_q = \{Q,1,2,3,4\}, V_s = \{5,S\}, \text{ capacity } K = 11$$

Question 124

a) + b)

		Iteration			
Node	Start solution	1	2	3	4
q	$(+,7)$	$(+,∞)$	$(+,∞)$	$(+,∞)$	$(+,∞)$
1	$(q^+,2),(2^+,1)$		$(2^+,1)$	$(2^+,2)$	
2	$(q^+,4)$		$(q^+,1)$	$(3^-,2)$	
3	$(q^+,1),(2^+,3)$	$(q^+,5)$		$(q^+,3)$	$(q^+,1)$
s	$(1^+,3),(3^+,4)$	$(3^+,2)$	$(1^+,1)$	$(1^+,2)$	
Δ	7	2	1	2	
Σ	7	9	10	12	

c)

$$V_q = \{q,3\}, V_s = \{1,2,s\}$$

$$\text{capacity } K = 2 + 5 + 6 - 1 = 12$$

5 Dynamic Programming

Question 125

- a)
- x_i : Number of additional scientists hired for team i
 - z_i : Number of additional scientists hired for the teams $1, \dots, i$.
 - $f_i(x_i)$: Probability of failure for team i if x_i additional scientists are hired for this team.
 - $F_i(z_i)$: Probability of failure for the teams $i + 1, \dots, 3$ if there are $1, \dots, i$ additional scientists hired for these teams in total.
 - $F_i^*(z_i)$: Minimal probability of failure for the teams $i + 1, \dots, 3$ if there are $1, \dots, i$ additional scientists hired for these teams in total.

b)

z_2	x_3	z_3	$f_3(x_3)$	$F_3^*(z_3)$	$F_2(z_2)$	
0	2	2	0.30	1	0.30	*
1	1	2	0.50	1	0.50	*
2	0	2	0.80	1	0.80	*

z_1	x_2	z_2	$f_2(x_2)$	$F_2^*(z_2)$	$F_1(z_1)$	
0	0	0	0.60	0.30	0.18	
0	1	1	0.40	0.50	0.20	
0	2	2	0.20	0.80	0.16	*
1	0	1	0.60	0.50	0.30	*
1	1	2	0.40	0.80	0.32	
2	0	2	0.60	0.80	0.48	*

z_0	x_1	z_1	$f_1(x_1)$	$F_1^*(z_1)$	$F_0(z_0)$	
0	0	0	0.40	0.16	0.064	
0	1	1	0.20	0.30	0.060	*
0	2	2	0.15	0.48	0.072	

The optimal solution is:

There is an additional scientist hired for the first and the third team respectively. The probability of success is 94%

Question 126

a)

z_1	x_1	z_0	$f_1(x_1)$	$F_0^*(z_0)$	$F_1(z_1)$	
0	0	0	0	0	0	*
3	1	0	19	0	19	*

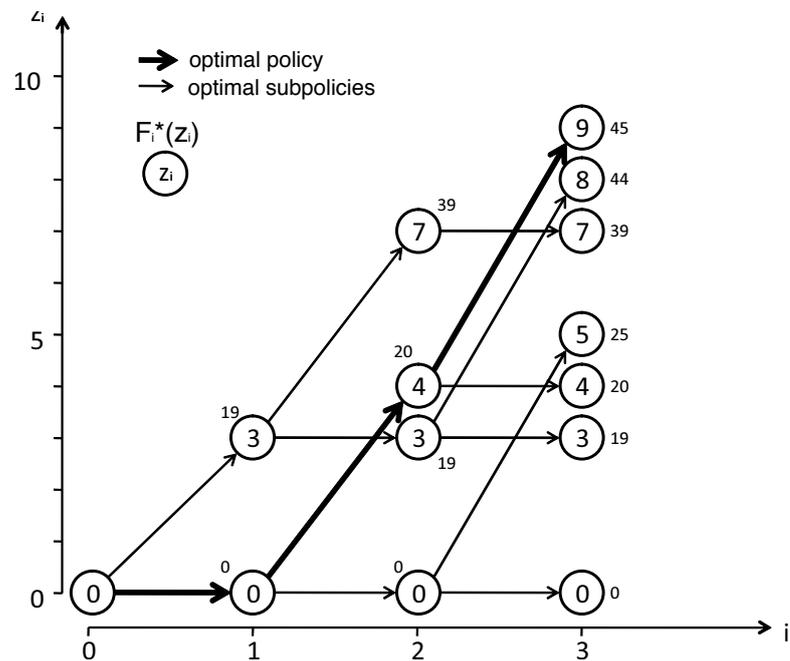
z_2	x_2	z_1	$f_2(x_2)$	$F_1^*(z_1)$	$F_2(z_2)$	
0	0	0	0	0	0	*
3	0	3	0	19	19	*
4	1	0	20	0	20	*
7	1	3	20	19	39	*

z_3	x_3	z_2	$f_3(x_3)$	$F_2^*(z_2)$	$F_3(z_3)$	
0	0	0	0	0	0	*
3	0	3	0	19	19	*
4	0	4	0	20	20	*
7	0	7	0	39	39	*
5	1	0	25	0	25	*
8	1	3	25	19	44	*
9	1	4	25	20	45	*

Hans invests his savings in alternatives 2 and 3, because he generates the highest objective value that way. The total expenses are 9.

b) The alternative ($z_2 = 7, x_3 = 1$) is not considered, because the budget restriction would not be satisfied. This investment alternative is not feasible.

c)



Question 127

- a)
- x_i : Number of days spent on chapter i .
 - z_i : Number of days spent on the chapters $1, \dots, i$.
 - $f(x_i)$: Points achieved for chapter i if x_i days are spent on chapter i .
 - $F_i(z_i)$: Points achieved in total for the chapters $0, \dots, i$ if z_i days are spent.
 - $F_i^*(z_i)$: Maximum points achieved in total for the chapters $0, \dots, i$ if z_i days are spent.

b)

z_1	x_1	z_0	$f_1(x_1)$	$F_0^*(z_0)$	$F_1(z_1)$	
2	2	0	5	0	5	*
3	3	0	7	0	7	*
4	4	0	7	0	7	*
5	5	0	9	0	9	*

z_2	x_2	z_1	$f_2(x_2)$	$F_1^*(z_1)$	$F_2(z_2)$	
4	2	2	4	5	9	*
5	3	2	6	5	11	*
6	4	2	8	5	13	*
7	5	2	9	5	14	
5	2	3	4	7	11	*
6	3	3	6	7	13	*
7	4	3	8	7	15	*
6	2	4	4	7	11	
7	3	4	6	7	13	
7	2	5	4	9	13	

z_3	x_3	z_2	$f_3(x_3)$	$F_2^*(z_2)$	$F_3(z_3)$	
6	2	4	6	9	15	*
7	3	4	7	9	16	
8	4	4	8	9	17	
9	5	4	9	9	18	
7	2	5	6	11	17	*
8	3	5	7	11	18	
9	4	5	8	11	19	
8	2	6	6	13	19	*
9	3	6	7	13	20	
9	2	7	6	15	21	*

z_4	x_4	z_3	$f_4(x_4)$	$F_3^*(z_3)$	$F_4(z_4)$
11	5	6	8	15	23
11	4	7	8	17	25
11	3	8	5	19	24
11	2	9	5	21	26 *

The optimal solution is:

Spend 3 days on the chapter Decision Theory, 4 days on the chapter Linear Programming, 2 days on the chapter Integer Programming and 2 days on the chapter Network Theory. The maximum number of points achieved are 26.

Question 128

- a)
- x_i : Amount ordered at the beginning of week i .
 - z_i : Storage at the end of week i .
 - $f_i(z_i, x_i)$: Costs in period i , if the storage at the end of week i is z_i and x_i is ordered at the beginning of week i .
 - $F_i(z_i)$: Costs for the weeks $i + 1, \dots, 3$, if at the end of week i the storage amounts to z_i .
 - $F_i^*(z_i)$: Minimum costs for the weeks $i + 1, \dots, 3$, if at the end of week i the storage amounts to z_i .

b)

z_2	x_3	z_3	$f_3(z_3, x_3)$	$F_3^*(z_3)$	$F_2(z_2)$
40	0	0	0	0	0 *
0	40	0	150	0	150 *

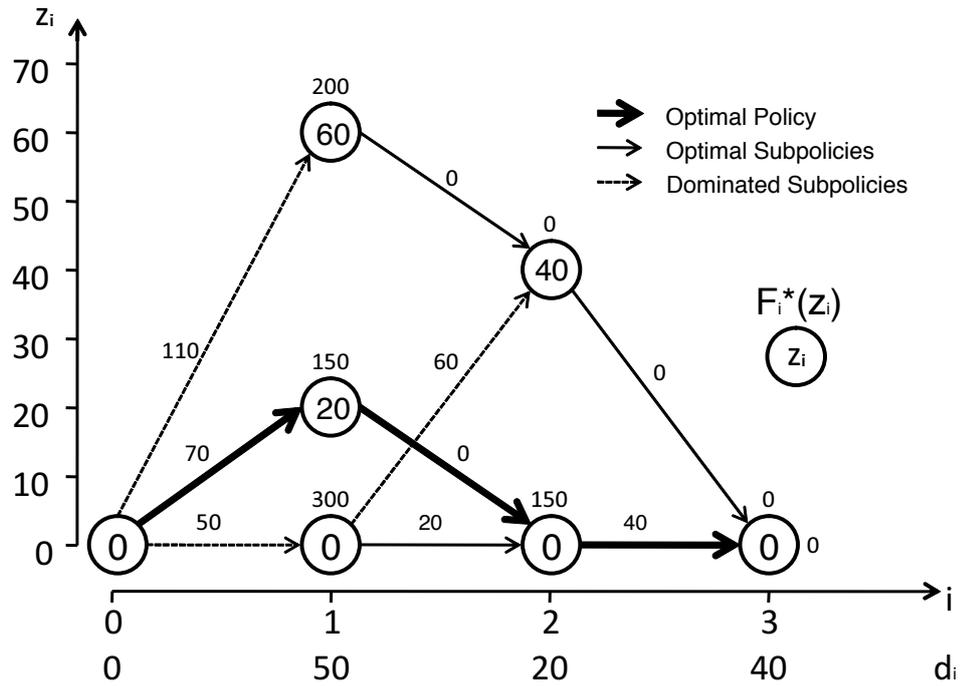
z_1	x_2	z_2	$f_2(z_2, x_2)$	$F_2^*(z_2)$	$F_1(z_1)$
20	0	0	0	150	150 *
0	20	0	150	150	300 *
60	0	40	200	0	200 *
0	60	40	350	0	350

z_0	x_1	z_1	$f_1(z_1, x_1)$	$F_1^*(z_1)$	$F_0(z_0)$
0	50	0	150	300	450
0	70	20	250	150	400 *
0	110	60	450	200	650

The optimal solution is:

Order at the beginning of period 1 the demand of the periods 1 and 2 (70 units of raw material) and at the beginning of period 3 the demand for period 3 (40 units of raw material). The minimum total cost is 300 Euro of fixed ordering cost plus 100 Euro of storage cost.

c)



Question 129

- a)
- x_i : Amount ordered at the beginning of month i .
 - z_i : Storage at the end of month i .
 - $f_i(z_i, x_i)$: Costs in period i , if the storage at the end of month i is z_i and x_i is ordered at the beginning of week i .
 - $F_i(z_i)$: Costs for the months $i + 1, \dots, 5$, if at the end of month i the storage amounts to z_i .
 - $F_i^*(z_i)$: Minimum costs for the months $i + 1, \dots, 5$, if at the end of month i the storage amounts to z_i .

b)

z_4	x_5	z_5	$f_5(z_5, x_5)$	$F_5^*(z_5)$	$F_4(z_4)$	
8,000	0	0	0	0	0	*
0	8,000	0	200	0	200	*

z_3	x_4	z_4	$f_4(z_4, x_4)$	$F_4^*(z_4)$	$F_3(z_3)$	
27,000	0	0	0	200	200	*
0	27,000	0	200	200	400	
35,000	0	8,000	48	0	48	*
0	35,000	8,000	248	0	248	*

z_2	x_3	z_3	$f_3(z_3, x_3)$	$F_3^*(z_3)$	$F_2(z_2)$	
22,000	0	0	0	248	248	*
0	22,000	0	200	248	448	*
49,000	0	27,000	162	200	362	*
0	49,000	27,000	362	200	562	
57,000	0	35,000	210	48	258	*
0	57,000	35,000	410	48	458	

z_1	x_2	z_2	$f_2(z_2, x_2)$	$F_2^*(z_2)$	$F_1(z_1)$	
4,000	0	0	0	448	448	*
0	4,000	0	200	448	648	
26,000	0	22,000	132	248	380	*
0	26,000	22,000	332	248	580	*
53,000	0	49,000	294	362	656	*
0	53,000	49,000	494	362	856	
61,000	0	57,000	342	258	600	*
0	61,000	57,000	542	258	800	

z_0	x_1	z_1	$f_1(z_1, x_1)$	$F_1^*(z_1)$	$F_0(z_0)$	
0	6,000	0	200	580	780	
0	10,000	4,000	224	448	672	*
0	32,000	26,000	356	380	736	
0	59,000	53,000	518	656	1174	
0	67,000	61,000	566	600	1166	

The optimal solution is:

Order the total demand of periods 1 and 2 (10,000 components) in period 1, the demand of period 3 (22,000 components) in period 3 and the total demand of periods 4 and 5 in period 4 (35,000 components). The minimum total costs are 600 Euro of fixed ordering cost plus 72 Euro of storage cost.

Question 130

- a)
- x_i : Financial value of securities, which are sold in period i .
 - z_i : Liquid assets gained from the sale of the securities available at the end of period i .
 - $f_i(z_i, x_i)$: Costs in period i , if there were z_i of liquid assets available at the end of period i and securities of a financial value of x_i are sold at the beginning of period i .
 - $F_i(z_i)$: Total costs for the periods $i + 1, \dots, 6$, if there are z_i of liquid assets available at the end of period i .

- $F_i^*(z_i)$: Minimum total costs for the periods $i + 1, \dots, 6$, if there are z_i of liquid assets available at the end of the previous period i .

b)

z_5	x_6	z_6	$f_6(z_6, x_6)$	$F_6^*(z_6)$	$F_5(z_5)$	
1,500	0	0	150	0	150	*
0	1,500	0	250	0	250	*

z_4	x_5	z_5	$f_5(z_5, x_5)$	$F_5^*(z_5)$	$F_4(z_4)$	
3,000	0	0	300	250	550	*
0	3,000	0	250	250	500	
4,500	0	1,500	450	150	600	*
0	4,500	1,500	250	150	400	*

z_3	x_4	z_4	$f_4(z_4, x_4)$	$F_4^*(z_4)$	$F_3(z_3)$	
1,000	0	0	100	400	500	*
0	1,000	0	250	400	650	*
4,000	0	3,000	400	550	950	*
0	4,000	3,000	250	550	800	
5,500	0	4,500	550	600	1,150	*
0	5,500	4,500	250	600	850	

z_2	x_3	z_3	$f_3(z_3, x_3)$	$F_3^*(z_3)$	$F_2(z_2)$	
500	0	0	50	650	700	*
0	500	0	250	650	900	
1,500	0	1,000	150	500	650	*
0	1,500	1,000	250	500	750	*
4,500	0	4,000	450	950	1,400	*
0	4,500	4,000	250	950	1,200	
6,000	0	5,500	600	1,150	1,750	*
0	6,000	5,500	250	1,150	1,400	

z_1	x_2	z_2	$f_2(z_2, x_2)$	$F_2^*(z_2)$	$F_1(z_1)$	
2,500	0	0	250	750	1,000	*
0	2,500	0	250	750	1,000	
3,000	0	500	300	700	1,000	*
0	3,000	500	250	700	950	
4,000	0	1,500	400	650	1,050	*
0	4,000	1,500	250	650	900	*
7,000	0	4,500	700	1,400	2,100	*
0	7,000	4,500	250	1,400	1,650	
8,500	0	6,000	850	1,750	2,600	*
0	8,500	6,000	250	1,750	2,000	

z_0	x_1	z_1	$f_1(z_1, x_1)$	$F_1^*(z_1)$	$F_0(z_0)$	
0	1,500	0	250	900	1,150	*
0	4,000	2,500	250	1,000	1,250	
0	4,500	3,000	250	1,000	1,250	
0	5,500	4,000	250	1,050	1,300	
0	8,500	7,000	250	2,100	2,350	
0	10,000	8,500	250	2,600	2,850	

The optimal solution is:

At the beginning of period 1 securities of the financial value of 1,500 Euro are sold, at the beginning of period 2 securities of the financial value of 4,000 Euro are sold and at the beginning of period 5 securities of the financial value of 4,500 Euro are sold. The minimum total costs are 1,150 Euro.

Question 131

- a)
- Stage 1 includes node 1.
 - Stage 2 includes nodes 2, 3 and 4.
 - Stage 3 includes nodes 5 and 6.
 - Stage 4 includes node 7.
- b)

z_3	x_3	z_4	$f_3(z_3, z_4)$	$F_4^*(z_4)$	$F_3(z_3)$	
5	(5,7)	7	2	0	2	*
6	(6,7)	7	4	0	4	*

z_2	x_2	z_3	$f_2(z_2, z_3)$	$F_3^*(z_3)$	$F_2(z_2)$	
2	(2,5)	5	6	2	8	*
3	(3,5)	5	2	2	4	*
3	(3,6)	6	2	4	6	
4	(4,6)	6	3	4	7	*

z_1	x_1	z_2	$f_1(z_1, z_2)$	$F_2^*(z_2)$	$F_1(z_1)$
1	(1,2)	2	5	8	13
1	(1,3)	3	3	4	7 *
1	(1,4)	4	4	7	11

The shortest path from node 1 to node 7 is (1,3,5,7) with a length of $F_1^*(z_1) = 7$.

Question 132

- a)
- x_i : Arc, which defines a path from one state in stage i to one state in stage $i + 1$.
 - z_i : Considered node (state) in stage i .
 - $f_i(z_i, z_{i+1})$: Distance from state z_i in stage i on arc x_i to stage z_{i+1} in stage $i + 1$.
 - $F_i(z_i)$: Path from node z_i on stage i to single node z_n on stage n .
 - $F_i^*(z_i)$: Shortest path from node z_i on stage i to single node z_n on stage n .

b)

z_4	x_4	z_5	$f_4(z_4, z_5)$	$F_5^*(z_5)$	$F_4(z_4)$
7	(7,8)	8	2	0	2 *

z_3	x_3	z_4	$f_3(z_3, z_4)$	$F_4^*(z_4)$	$F_3(z_3)$
4	(4,7)	7	2	2	4 *
5	(5,7)	7	5	2	7 *
6	(6,7)	7	4	2	6 *

z_2	x_2	z_3	$f_2(z_2, z_3)$	$F_3^*(z_3)$	$F_2(z_2)$
2	(2,4)	4	3	4	7 *
2	(2,5)	5	4	7	11
2	(2,6)	6	2	6	8
3	(3,4)	4	4	4	8 *
3	(3,5)	5	1	7	8 *
3	(3,6)	6	3	6	9

z_1	x_1	z_2	$f_1(z_1, z_2)$	$F_2^*(z_2)$	$F_1(z_1)$
1	(1,2)	2	3	7	10 *
1	(1,3)	3	3	8	11

The shortest path from node 1 to node 8 is (1,2,4,7,8) with length $F_1^*(z_1) = 10$.