Exercise Appendix

Operations Research and Decision Analysis

Winter Semester 2025/2026

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Contents

= maybe do later

1 Decision Theory

1.1 Decision Context

1.1.1 Basics

Question 1
Name and explain the criteria that describe different decision contexts as explained in the lecture.

Level of uncertainty
No. of objectives
No. of decision makers
Opponent behavior

Question 2
A small investor considers investing his savings of 8,000 Euro into securities for one year. He has two investment options to choose from:

- Buying fixed-income securities at a price of 3,000 Euro per unit with an interest coupon of 10% per year.
- Buying shares of an equity mutual fund at a price of 4,000 Euro per share. If the stock market develops positively, the share value will increase by 20% in a year. Otherwise, it will decrease by 20%.

a) Explain these terms using this example:
   (1) Alternative investment option
   (2) The set of alternatives set of all investment options
   (3) States of nature (or scenarios) positive/negative stock market development

b) What alternatives does the investor have?

c) Construct a decision matrix. → see c)

c)

Alt.	€ bonds	€ fund	stock+	stock−
1	3K	/	8,3K	8,3K
2	6K	/	8,6K	8,6K
3	/	4K	8,8K	7,2K
4	/	8K	9,6K	6,6K
5	3K	4K	9,1K	7,5K
0	/	/	8K	8K

1.1.2 Efficient alternatives

Question 3
Consider the following decision matrix for a maximization problem:

	s_1	s_2	s_3	s_4	s_5
a_1	13	21	25	18	10
a_2	14	15	20	14	18
a_3	11	18	24	13	13
a_4	12	19	25	13	x

→ $x \in (11, 12)$
$x < 13$ to not dominate a_3
$x > 10$ to not be dominated by a_1

Determine the integer value(s) for $x \in \mathbb{N}$ such that all alternatives are efficient or non-dominated.

Question 4
Consider the following decision matrix for a maximization problem:

	s_1	s_2	s_3	s_4	s_5	s_6
$\cancel{a_1}$	−1	0	2	1	3	2
$\cancel{a_2}$	−2	0	1	−1	2	−1
a_3	−1	1	2	1	3	−2
a_4	1	0	2	−1	4	2
a_5	0	1	2	1	3	−2
$\cancel{a_6}$	−1	0	1	−1	3	−1

→ and a_4, a_6
a_1 dominates a_2
→ and a_1
a_4 dominates a_6
a_5 dominates a_3

a) Which alternatives are efficient and which are inefficient or dominated?

b) Reconstruct the decision matrix by removing all dominated alternatives.

1.2 Decision Under Uncertainty

1.2.1 Decision rules

Question 5 bro what

Characterize the decision contexts in which the decision rules can be used to solve a decision problem.

Question 6

Describe the decision rules that were presented in the lecture. State the decision maker's attitude toward uncertainty that is reflected by each decision rule applied.

Maximax
Maximin
Hurwicz
Minimax Regret
Laplace

Question 7

Consider the following decision matrix of a maximization problem:

max regret		s_1	s_2	s_3	s_4
5	a_1	-1	5	0	3
9	a_2	4	-2	1	2
6	a_3	3	7	1	-3
8	a_4	1	-1	0	0

a) Remove all dominated alternatives, if any, and reconstruct the decision matrix to include only the efficient alternatives. None dominated

b) Which alternative is chosen under each of the following decision rules?

(1) Maximax rule a_3

(2) Maximin rule a_1

(3) Laplace rule a_3

(4) Minimax Regret rule a_1

(5) Hurwicz rule; $\lambda_1 = 0.25$, $\lambda_2 = 0.75$ nah thank you

$\searrow \frac{1}{4}$ min + $\frac{3}{4}$ max

Question 8

A tour operator has the chance to buy 0, 1000, 2000 or 3000 nights for 90 Euro per night in a hotel in Antalya for the next season. If he sells a night within one of his package holidays, he would earn 100 Euro for that night. He estimates to be able to sell 1800 nights in an average year and 900 (3000) nights in a bad (good) year.

a	bought (#)	good (k€)	avg (k€)	bad (k€)
0	0	0	0	0
1	1000	+10	+10	+9 − 9 = 0
2	2000	+20	+18 − 18 = 0	+9 − 99 = −90
3	3000	+30	+18 − 108 = −90	+9 − 189 = −180

a) Construct a decision matrix.

b) Remove all dominated alternatives, if any, and reconstruct the decision matrix to include only the efficient alternatives.

c) How many nights should the tour operator buy, if he decides under one of the following rules?

(1) Maximax rule 3000

(2) Maximin rule 1000

(3) Laplace rule 1000

(4) Minimax Regret rule 1000

Question 9

Rosemarie sells red roses in Indian restaurants in order to earn some extra money. She estimates to sell either zero, one, two, three or even four bunches of roses for 8 Euro each in one evening. Unfortunately, she has no idea about the probabilities of these estimates. Each bunch of roses costs her 6 Euro and has no value if not sold.

	S_0	S_1	S_2	S_3	S_4	Ø	max regret
a_0	—	—	—	—	—	0	8
a_1	-6	2	=	=	=	0.4	6
a_2	-12	-4	4	=	=	-0.8	12
a_3	-18	-10	-2	6	=	-3.6	18
a_4	-24	-16	-8	0	8	-8	24

a) Construct a decision matrix.

b) Remove all dominated alternatives, if any, and reconstruct the decision matrix to include only the efficient alternatives.

c) How many bunches of roses should Rosemarie buy if she makes her decision under one of the following rules?

(1) Maximax-Regel 4

(2) Maximin-Regel 0

(3) Laplace-Regel 1

(4) Minimax Regret-Regel 1

Question 10

minimize cost!

The owner of a drilling concession for an oilfield wants to conduct a test drilling. There are three different methods a_i ($i = 1, 2, 3$) to conduct this test. The incurred costs of each method depends on the uncertain kind of rock in this area. Experienced geologists believe in the occurrence of 5 different kinds of rock s_j ($j = 1, 2, \dots, 5$). The incurred costs of each method depending on each kind of rock are listed below:

	s_1	s_2	s_3	s_4	s_5	max regret
a_1	100,000	100,000	150,000	250,000	300,000	100k
a_2	50,000	200,000	100,000	250,000	300,000	100k
a_3	200,000	150,000	150,000	200,000	200,000	50k

Which method does the owner choose if he decides under the Minimax Regret rule? a_1 or a_2

1.3 Decision Under Risk

1.3.1 Expected value criterion

Question 11

When can the expected value criterion be used for solving a decision problem? Characterize the decision context.

Question 12

A risk-neutral art collector considers securing his recently acquired paintings by Pablo Picasso amounting to 20 million Euro against theft. The insurance company offers him a theft insurance for the premium of 3,750 per year. The probability of theft within the next year is known to be 10^{-4}.

a) Construct decision matrix from the art collector's point of view.

b) Formulate the two alternatives into two lotteries.

c) Which alternative will the art collector choose?

a_0 no insurance s_0 not stolen (0.9999)

a_1 insurance s_1 stolen (0.0001)

	s_0	s_1
a_0	0	-20'000'000
a_1	-3750	-3750

L_0: -20 M / 0

L_1 — -3750

$L_0 = -2000$

$L_1 = -3750$

→ no insurance

1.3.2 Expected utility criterion

Question 13

When can the expected utility criterion be used for solving a decision problem? Characterize the decision context. Same as expected value, but DM has risk attitude (averse or taking)

Question 14

Name and explain the axioms that must be satisfied by a decision maker, if he wants to use the expected utility criterion to evaluate the lotteries.

Question 15

The utility function of Mr. Hinz is estimated as $u(x) = \frac{1}{2} + (\frac{x}{20})^2$. He is offered to participate in a lottery, which brings a payout of 10 Euro with a probability of 80%. Otherwise the payout is 3.

Inverse u to get € for EU

$EV = 10(0.8)+3(0.2) = 8.6$

$EU = \left(\frac{1}{2}+\left(\frac{10}{20}\right)^2\right)(0.8)+\left(\frac{1}{2}+\left(\frac{3}{20}\right)^2\right)(0.2) = 0.7045$

$CE = u^{-1}(EU) = 20\cdot\sqrt{0.7045-0.5} = 9.044$

$RP = EV - CE = 8.6 - 9.044 = -0.444$

a) Determine the expected utility, the certainty equivalent and the risk premium.

b) Which attitude toward risk does Mr. Hinz have? He's paying 0.444€ more than risk-neutrals therefore: Risk-Seeking (RP<0)

Question 16
The utility function of Mr. Kunz is estimated as $u(x) = \sqrt{\frac{x}{10}}$. He is offered to participate in a lottery, which brings a payout of 10 Euro with a probability of 64%. Otherwise the payout is 0.

a) Determine the expected utility, the certainty equivalent and the risk premium.

b) Which attitude toward risk does Mr. Hinz have?

c) Which amount would Mr. Kunz or a risk-neutral decision maker pay at most to participate in the lottery?

Question 17
The utility function of one decision maker is estimated as $u(x) = (\frac{x}{100})^2$. He is offered to participate in a lottery which brings a payout of 100 Euro with a probability of 25% and a payout of 0 with a probability of 75%.

$EV = 25$; $EU = 0.25(\frac{100}{100}{}^2)+0 = 0.25$; $CE = 100\cdot\sqrt{0.25} = 50$; $RP = EV - CE = -25$

a) Determine the expected utility, the certainty equivalent and the risk premium.

b) Which attitude toward risk does the decision maker have? RP<0 → Risk-Seeking

c) There is another lottery which brings a payout of 80 Euro with a probability of p and a payout of 20, otherwise. For which probability p would the decision maker be indifferent between the two lotteries? $EU_1 = 0.25$

$EU_2 = p\left(\frac{80}{100}\right)^2+(1-p)\left(\frac{20}{100}\right)^2 = 0.64p+0.04-0.04p = 0.6p+0.04$

$EU_1 = EU_2 \Rightarrow 0.6p = 0.21$
$\Rightarrow p = 0.35$

Question 18
Consider a lottery which brings a payout of 100 Euro with a probability of 20%, a payout of 50 Euro with a probability 50% and a payout of 0 with a probability of 30%.

a) Assume that the certainty equivalent of a decision maker is 50 Euro. Which risk attitude does the decision maker have? $EV = 20+25+0 = 45$; $RP = EV - CE = -5$ → RP<0 → Risk-Seeking

b) Another decision maker has his utility function estimated as $u(x) = \frac{1}{10}\cdot\sqrt{x}$. Determine the amount y, for which he is indifferent between the above lottery and a lottery which brings a payout of either y Euro or $\frac{y}{4}$ Euro, each with a probability of 50%.

$EU_1 = 20\%\cdot u(100)+50\%\cdot u(50) \approx 0.55$

$EU_1 = EU_2 \Rightarrow 0.55 = 0.05\sqrt{y}+0.05\sqrt{y/4} \rightarrow 11 = \sqrt{y}+\frac{1}{2}\sqrt{y} \Rightarrow 22 = 3\sqrt{y} \Rightarrow y = (\frac{22}{3})^2 = 53.78$

Question 19
A businessman has the utility function $u(x) = \frac{1}{2}\cdot\left(\frac{x}{100{,}000}\right)^2 + \frac{x}{200{,}000}$. He is offered two projects. The first project delivers an income of either 20,000 Euro or 40,000 Euro, each with a probability of 50% . The second project brings an income of y Euro with a probability of 70% and 0 Euro, otherwise. Determine the income y of the second project, for which he is indifferent between the two projects.

$EU_1 = 0.2$
$EU_2 = 0.7\left(0.5\left(\frac{y}{100000}\right)^2+\frac{y}{200000}\right)$

$EU_1 = EU_2 \Rightarrow 0.57 = \left(\frac{y}{100000}\right)^2+\frac{y}{100000}$
$57142 = \frac{1}{100000}y^2+y$

Question 20
The football player Rheuma Kai has the choice between two contracts. The first contract guarantees a safe annual salary of 240,000 Euro. The second one pays based on his performance. If his team wins the championship with a probability of 30%, he will receive 450,000 Euro, or otherwise only 150,000 Euro. His utility function is estimated as: (piecewise defined)

$$u(x) = \begin{cases} \frac{1}{1{,}000}\sqrt{x} & \text{for} \quad 0 \le x \le 250{,}000 \\ \frac{1}{2} + \left(\frac{x-250{,}000}{396{,}850}\right)^{\frac{3}{2}} & \text{for} \quad 250{,}000 < x \le 500{,}000 \end{cases}$$

a) How high is his expected salary in both cases? 240'000 € each

b) Which contract will he choose? $EU_1 = 0.49$; $EU_2 = 0.3\cdot u(450000)+0.7\cdot u(150000) = 0.53$ → Second contract

c) How high should the safe salary be, so that he will choose against the performance-based contract?

$0.53 = u(x) \Rightarrow x = u^{-1}(0.53) = 0.03^{2/3}\cdot 396850+250000 = \underline{288315.45}$

Question 21
A decision maker's preference is described by the utility function $u(x) = x^a$ (with $x \geq 0$ und $a > 0$).

a) Use the Arrow-Pratt measure to specify for which values of a the decision maker is

(1) Risk-neutral $a = 1$

(2) Risk-averse $a < 1$

(3) Risk-seeking $a > 1$

Arrow-Pratt: $-\frac{u''(x)}{u'(x)} = -\frac{(a^2-a)x^{a-2}}{ax^{a-1}} = -\frac{(a-1)x^{a-2}}{x^{a-1}} = -\frac{a-1}{x} = \frac{1-a}{x}$

$x^{(a-2)-(a-1)=-1}$

Seeking < 0 < averse

b) The decision maker has the opportunity to participate in a lottery which brings a payout of either 1 Euro or 0 Euro, each with a probability of 50%.

$EU = \frac{1}{2} 1^a = \frac{1}{2}$

(1) How high is the certainty equivalent of the decision maker depending on a? $CE = u^{-1}(\frac{1}{2}) = 0.5^{\frac{1}{a}}$

(2) Assume that $a = \frac{3}{4}$. Will the decision maker pay 30 cents for his/her participation on the lottery? $CE = 0.5^{4/3} = 0.397$ → Yes, would pay up to CE = 39 cents

Question 22
The managing director of a medium-sized company is facing with the decision whether to include a new product into the assortment. In the end he has only products A and B in his shortlist, but only one can be selected. Product A brings a profit of 50,000 Euro with a probability of 30%, a profit of 90,000 Euro with a probability of 50% and a profit of 100,000 Euro with a probability of 20% in the planning period. Product B brings a profit of 80,000 Euro in any case. The managing director asked two consultants, Mr. Roland and Mr. Berger, to make a decision. Mr. Rolands has the utility function

$$u(x) = \frac{x}{100,000}$$

whereas Mr. Berger has the following (piecewise-defined) utility function

$$v(x) = \begin{cases} \frac{1}{2} \cdot \left(\frac{x}{50,000}\right)^2 & \text{for} \quad 0 \leq x \leq 50,000 \\ \frac{1}{2} + \sqrt{\frac{x-50,000}{200,000}} & \text{for} \quad 50,000 < x \leq 100,000 \end{cases}$$

a) Roland: $-\frac{0}{1/100000} = 0$ → neutral

Berger: for $0 \leq x \leq 50000$

$-\frac{1/50000^2}{x/50000^2} = -\frac{1}{x}$ → Seeking

for $50000 \leq x \leq 10000$

$-\frac{-\frac{1}{4}\left(\frac{x-50000}{200000}\right)^{-3/2} \cdot \left(\frac{1}{200000}\right)^2}{\frac{1}{2}\left(\frac{x-50000}{200000}\right)^{-1/2} \cdot \frac{1}{200000}}$

$= \frac{1}{2}\left(\frac{x-50000}{200000}\right)^{-1} \cdot \frac{1}{200000}$

$= \frac{1}{2(x-50000)}$ → Avoiding

a) Determine the risk attitude of the two consultants by using the Arrow-Pratt measure.

b) What are the recommendations of the consultants?

Roland: $EU_1 = 0.8$; $EU_2 = 0.8$ → neutral

Berger: $EU_1 = 0.3 \cdot 0.5 + 0.5 \cdot 0.85 + 0.2 \cdot 1 = 0.83$; $EU_2 = 0.85$ → Product B

c) How do the decisions change, if both products incur an additional fixed costs of 50,000 Euro and how can this effect be explained? Rolands decision doesn't change.

Berger: $EU_1 = 0.3 \cdot 0 + 0.5 \cdot 0.32 + 0.2 \cdot 0.5 = 0.26$

$EU_2 = 0.18$ → Berger changes to Product A bc. of the piecewise utility

1.3.3 Decision matrix and risk utility

Question 23
The ChraimlerDysler AG wants to secure its position in the American automobile market and plans a new factory in Detroit. The corporate management considers producing either the series Tiger, Eagle or Dolphin. The following matrix, which illustrates the expected income of each alternative depending on the situation of the total market, is used as a decision support.

	$p_1 = 0.2$	$p_2 = 0.5$	$p_3 = 0.3$	EV
Total market	positive	constant	negative	
Tiger	70	20	5	25.5
Eagle	40	30	25	30.5
Dolphin	10	30	50	32

a) Which alternative will the corporate management choose, if they use only the expected value as the decision criterion? Which risk attitude is implied by the corporate management?

Dolphin. Risk-neutral because only EV is used; basically $u(x)=x$

b) Assume that the corporate management has the risk utility function $u(x) = \left(\frac{x-5}{65}\right)^2$. Which decision would be rational if they use this risk utility function? $EU_{Tiger} = 0.2(1)+0.5(0.053)+0.3(0) = 0.23$
$EU_{Eagle} = 0.16$
$EU_{Dolphin} = 0.22$ → Tiger

Question 24

A publicly traded company requires additional capital to finance an investment of 100 million Euro. The supervisory board offers the management to select either to take out a loan, to issue a bond or to issue new stocks. The following decision matrix, which illustrates the expected costs in millions Euro, depending on the development of the interest rate, is used as a decision support.

	$0 \leq p_1 \leq 0.8$	$p_2 = 0.2$	$p_3 = 0.8 - p_1$
Interest Rate	increase	constant	decrease
Loan	6.3	4.5	3.0
Bond	5.1	4.8	3.9
Stocks	3.5	4.7	5.6

a) Which criterion will the management use, if they are risk-neutral? ~~Using min(EV)~~ No result! Just "Expected Value Criterion"

b) Assume that the management acts on the utility function $u(x) = \left(\frac{6.3-x}{3.3}\right)^2$.

(1) For which probability p_1 would the decision maker be indifferent between the borrowing and the issuing of stocks? $EU_L = 0.2(0.298) + (0.8-p_1)(1) = 0.86 - p_1$; $EU_S = p_1(0.72) + 0.2(0.23) + (0.8-p_1)(0.04) = 0.08 + 0.68\,p_1$

(2) For which probability p_1 would the decision maker choose the bond as a financing tool? $\Rightarrow p_1 = 0.46$
$EU_B = p_1(0.13) + 0.2(0.21) + (0.8-p_1)(0.53) = 0.466 - 0.4p_1$ To choose against EU_B: $p_1 > 0.65$
EU_S: $p_1 < 0.36$ → for no p_1; never!

1.3.4 Decision matrix and μ–σ rule

Question 25

In which context is the μ–σ rule used to solve a decision problem? Characterize the decision context.

Question 26

The student M. S. wants to invest his savings of 100 Euro. He has to choose between three different alternatives:

A: Invest in the company "Gugel" newly founded by one of his fellow students.

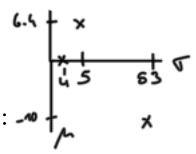
B: Purchase the stocks of the WMB AG.

C: Purchase the German government bonds.

The expected payoffs after one year are listed below:

	$p_1 = 0.2$	$p_2 = 0.7$	$p_3 = 0.1$	EV (μ)	Var (σ^2)	SD (σ)
	s_1	s_2	s_3			
A	-100	0	100	-10	$0.2(-100-(-10))^2 + 0.7(0-(-10))^2 + 0.1(100-(-10))^2 = 2900$	53.85
B	-4	8	16	6.4	32.64	5.71
C	4	4	4	4	0	0

a) Determine the expected outcome and the variance of each alternative.



b) Depict each alternative in a μ–σ–diagram.

c) Assume that M. S. behaves after one of the following preference functions:

i) $\Phi(a_i) = \mu(a_i) + \sigma(a_i)$. → Seeking, A

ii) $\Phi(a_i) = \mu(a_i)$. → Neutral, B

iii) $\Phi(a_i) = \mu(a_i) - \sigma(a_i)$. → Averse, C

(1) What is his attitude toward risk under each preference function?
(2) Which alternative would he prefer under each preference function?

Question 27
Mr. Bert wants to invest his savings of 100 Euro in securities for one year. There are three different alternatives available, each at the current market price of 100 Euro. The expected prices after one year are listed in the table below:

	$p_1 = 0.2$	$p_2 = 0.5$	$p_3 = 0.2$	$p_4 = 0.1$	μ	σ^2	σ
	s_1	s_2	s_3	s_4			
Security A	80	120	130	90	102	430	20.74
Security B	110	120	100	120	102	208	14.42
Security C	90	120	115	80	101	288	16.97

Somehow typed in the wrong numbers in the calculator, but result for c) is still correct

a) Determine the expected outcome and the variance of each alternative.

b) Depict each alternative in a μ–σ–diagram.

c) Assume that Mr. Bert behaves after the preference function $\Phi(a_i) = \frac{3}{2} \cdot \mu(a_i) + 2 \cdot \sigma(a_i)$.

(1) What is Mr. Bert's attitude toward risk? Seeking

(2) Which alternative would Mr. Bert prefer?

$u(a_1) = 194.68$
$u(a_2) = 181.84$ -> Security A
$u(a_3) = 185.44$

Question 28
Explain under which circumstances both the μ–σ and the expected utility rules provide an equal solution.
idk. ① Risk-Neutral DM ② Quadratic utility fn ③ Normally distributed results

Question 29
There are 300 Euro available for an investment of one year. Moreover, there are three different financial securities that fulfill the requirements. Securities A, B and C are at the current market prices of 100, 100 and 200 Euro per unit, respectively. The expected prices after one year are listed in the table below:

P		$p_1 = 0.2$	$p_2 = 0.5$	$p_3 = 0.1$	$p_4 = 0.2$	μ	σ^2	σ	Φ
		s_1	s_2	s_3	s_4				
100	Security A	80	120	130	90	107	341	18.47	2133.18
100	Security B	110	120	100	80	108	236	15.36	2155.28
200	Security C	180	270	300	140	229	3409	58.39	4511.82

a) Determine the expected outcome and the variance of each alternative.

b) Depict each security in a μ–σ–diagram.

c) Assume that the decision maker behaves after the preference function $\Phi(a_i) = 20 \cdot \mu(a_i) - 0.02 \cdot \sigma^2(a_i)$.

(1) What is the decision maker's attitude toward risk? Averse ($-\sigma$)

(2) Determine an optimal investment portfolio, assuming that each security cannot be invested in fractions. C has largest Φ/p ratio → Use as many C as possible: 1, and 1 of B (next best)
1C+1B

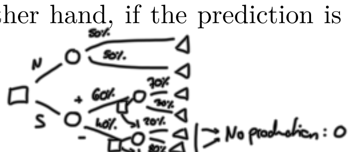
1.4 Multi-Stage Decision Making

1.4.1 Decision tree analysis and expected value criterion

Question 30
In which context can a decision tree be used to solve a decision problem? Characterize the decision context.

Question 31
A risk-neutral entrepreneur considers bringing a new product to market. The production of the new product will incur the fixed costs of 1.5 million Euro. If the product is a success (probability 50%), the entrepreneur will make 3 million Euro. But if the product is not a success, the entrepreneur will only make 1 million Euro.

The entrepreneur, however, has the opportunity to do a market survey for a cost of 20,000 Euro, before he finally makes the investment decision. If the market survey's prediction is positive (probability 60%), the probability that the product is a success will raise to 70%. On the other hand, if the prediction is negative, the probability that the product is a success declines to 20%.

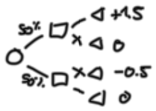


a) Draw a decision tree that depicts the problem of the entrepreneur.

b) Determine the optimal strategy for the decision maker. Which decisions should he make and how should he behave under certain situations? $EP_{S,+} = 2.4M - 1.5M = 0.9M$ $EP_N = 2M - 1.5M = 0.5M$ → $EP_N < EP_S$ ⇒ Do the survey
$EP_{S,-} = 1.4M - 1.5M = -0.1M$ → No production ⇒ $EP_{S,-} = 0$ $EP_S = 0.6 \cdot 0.9M - 0.02M = 0.54M$ if $EP_{S,-}$, don't produce

c) How much should the entrepreneur be willing to pay at the most for the information brought by the market survey? $0.6 \cdot 0.9M - c = 0.5M \Rightarrow c = 0.04M = 40000$ [€]

d) How much should the entrepreneur be willing to pay at the most for the perfect information that tells him whether his investment will be a success or not before he even makes the investment?



$EV = \frac{1}{2}(1.5M) + \frac{1}{2}(0) = 0.75M$; before: $EP_N = 0.5M \rightarrow c = 0.25M$

1.4.2 Decision tree analysis and risk utility

Question 32
The owner of a discontinued drilling concession has just enough time to do one more drilling. The drilling costs 100,000 Euro. The success probability in finding oil is 55% and he will earn 400,000 Euro. Otherwise, he will earn nothing. The owner, however, has the opportunity to perform a seismic test before the actual drilling. The test costs 30,000 Euro and the probability that the test is positive is 60%. Moreover, the chance to find oil raises to 85% if the test is positive. If the test is negative, however, the probability to find oil declines to 10%. The owner behaves after the risk utility function $u(x) = \left(\frac{x+130{,}000}{430{,}000}\right)^{\frac{2}{3}}$.

a) Draw a decision tree that depicts the owner's problem. Next page

b) Determine the optimal strategy for the decision maker. Which decisions should he make and how should he behave under certain situations? Do the test. If positive, drill, if not, don't drill.

c) How high may the probability of the situation „Seismic test is positive" be at the most, so that the owner does not decide to perform the seismic test? $0.63 > p \cdot (0.85 \cdot u(400k - 100k - 30k) + 0.15(u(-100k - 30k)) + (1-p) \cdot u(-30k)$
$> p \cdot 0.81 + (1-p) 0.38 \Rightarrow 0.25 > 0.43p \Rightarrow 0.58 > p$

Question 33
An entrepreneur plans the production of a new product that generates a return of 4 Euro. He estimates a high demand (87,000 units) with a probability of 60% and a low demand (45,000 units) with a probability of 40% in the first year. If the demand of the first year is high, he estimates a demand of 125,000 units with a probability of 70% and a demand of 79,000 units with a probability of 30% in the second year. However, if the demand of the first year is low, he estimates a demand of 58,000 units with a probability of 20% and a demand of 37,000 units with a probability of 80% in the second year. He already owns a machine that can produce 50,000 units annually. Moreover, he has the opportunity to buy a new machine at costs of 120,000 Euro that can produce 40,000 units annually.

Assume that the entrepreneur behaves after the risk utility function $u(x) = \sqrt{\frac{x}{1{,}000{,}000}}$.

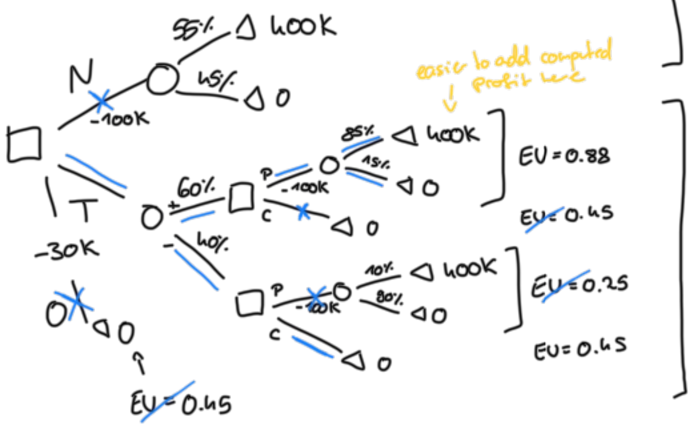
a) Draw the decision tree that depicts the problem of the entrepreneur.

b) Determine the optimal strategy for the decision maker. Which decisions should he make and how should he behave under certain situations?

1.4.3 Decision tree analysis and real options

Question 34

32, a)



$\cancel{EU} = 0.55 \cdot u(400k-100k) + 0.45 \cdot u(-100k)$
$= 0.6262$

$EU = 0.6(0.85(u(400k-100k-30k)) + 0.15(u(-100k-30k)))$
$+ 0.4(u(-30k)) = 0.6373$

34

A toy manufacturer plans the production of the new action figure „Terminator“. He estimates a high demand (154,000 units) with a probability of 38% and a low demand (68,000 units) with a probability of 62% in the first year. If the demand of the first year is high, he estimates a demand of 217,000 units with a probability of 73% and a demand of 171,000 units with a probability of 27% in the second year. However, if the demand of the first year is low, he estimates a demand of 166,000 units with a probability of 29% and a demand of 84,000 units with a probability of 71% in the second year. The management also considers when to buy which machines. There are two different types of machines available:

- The purchase of a machine of type I with an annual capacity of 250,000 units costs 527,315 Euro.
- The purchase of a machine of type II with an annual capacity of 120,000 units costs 264,141 Euro.

Assume that every sold action figure generates a revenue of 6 Euro. The discount interest rate is 12% annually.

a) Draw the decision tree that depicts the problem of the entrepreneur.

b) Determine the optimal strategy for the decision maker. Which decisions should he make and how should he behave under certain situations?

c) Determine the value of the real option for the following case: First, one machine of type II is purchased. Depending on the market development, an additional machine of type II is purchased.

1.5 Multi-Criteria Decision Making - Utility Analysis

Question 35
When can a utility analysis be used to solve a decision problem? Characterize the decision context.

Question 36
Describe the steps of a utility analysis in keywords.

Question 37
Three projects are suggested to the management of a company, but only one can be realized. The projects are evaluated on the basis of four criteria. The following table illustrates the evaluations of the projects and the weights of the criteria: a) b)

Criteria	Costs	Risk	Increased Market Share	Prestige	EU
Weight	6	8	4	2	
Project A	50,000 0	Low 1	5% 0.5	Medium 0	0.3(0)+0.4(1)+0.2(0.5)+0.1(0) =0.5
Project B	35,000 1	High 0	4% 0	Medium 0	= 0.3
Project C	48,000 2/15 = 0.13	Medium 0.5	6% 1	High 1	0.3(0.13)+0.2+0.2+0.1 =0.539
relative weight	0.3	0.4	0.2	0.1	

a) Conduct a utility analysis to decide which project the management should select. → Project C

b) Which weight should the goal „Increased market share “ have, so that project A will be selected, given that all other parameters are unchanged? C vs A: $0.339+\frac{w}{20}(1) < 0.4+\frac{w}{20}(0.5) \Rightarrow \frac{w}{20} < 0.122$
$\Rightarrow w < 2.44$

Question 38
A company wants to respond to the introduction of a new product by a competitor. There are three product alternatives A, B and C, which differ regarding the timing of the market introduction, the expected market share and the development costs. An early market introduction is three times as important to the company as the market share. The development costs are irrelevant. The company is risk-neutral.

Product	Market Introduction	Market Share	Development Costs
A	early	4%	300,000
B	middle	10%	400,000
C	late	8%	100,000

a) Conduct a utility analysis to decide which project the company should select.

b) Which weight should the market share have, so that the company will be indifferent between products A and B, given that all other parameters are unchanged?

c) Assume that the company applies the weight obtained in b), which product will be selected, if the expected market share of product B is only 9%? Will the decision change if the company is risk-seeking?

Question 39
You are a student assistant at a Chair of the TU Munich and supposed to recommend which word processor to be acquired. Since the Chair must pay for the software, the cost plays a crucial role ($g_1 = 5$). But you also want a program that is easy to use ($g_2 = 3$) and has a wide range of functions ($g_3 = 2$). After an intensive market analysis, you can call seven potential softwares and evaluate them for costs, usability and functionality.

	w	0.5	0.3	0.2		
b) eliminated	Software	Costs	Usability	Functions	S (utility value)	S, excl. g_2
X^F	Super Type	100 0	very difficult 0	many 1	0.2	
X^K	Easywrite	90 0.1	difficult 0.25	medium 0.66	0.26	
X^K	Right Word	50 0.5	very easy 1	medium 0.66	0.68	
	Lexico	0 1	medium 0.5	litte 0.33	0.72	$g_1+2\cdot 0.33$
X^L	Ultraword	20 0.8	very easy 1	very little 0	0.7	
	Keywrite	40 0.6	easy 0.75	medium 0.66	0.66	$g_1\cdot 0.6+2\cdot 0.66$
	Fastwrite	85 0.15	medium 0.5	many 1	0.425	$g_1\cdot 0.15+2$

a) Conduct a utility analysis to decide which project you should select. $\Rightarrow$ Lexico

b) The Chair is very experienced in using word processors. Therefore, the usability of the program can be ignored ($g_2 = 0$). Which weight should the criterion „Costs" have, so that „Keywrite" will be selected, given that all other parameters are unchanged? $K>L \cap K>F$

$0.6g_1+1.32 > g_1+0.66 \Rightarrow 1.65 > g_1$

$0.6g_1+1.32 > 0.15g_1+2 \Rightarrow g_1 > 1.51$

$\rightarrow 1.51 < g_1 < 1.65$

$0.43 < w_1 < 0.45$

1.6 Monte Carlo Simulation

Question 40
Describe the process of the Monte Carlo simulation. What information are available after the process of the simulation?

Question 41
The owner of a kiosk buys 13 copies of the Munich Daily Telegraph every day for 1.20 Euro each. He sells these for 1.50 Euro each. The copies that he cannot sell are worthless at the end of the day. The distribution of the daily demand is shown in the table below:

Demand	10	11	12	13	14	15	16
Probability	5%	10%	20%	25%	15%	15%	10%

The owner of the kiosk wants to maximize his profits.

a) Show the relationship between the input data and the resulting quantity in a deterministic model.

b) Which input data is stochastic and how is it distributed?

c) Perform a Monte Carlo simulation with 100 runs and determine the expected value and the standard deviation of the results.

Question 42
The airline LionAir links two regional airports. The airline owns a small turboprop aircraft that has a maximum capacity of 19 passengers. The following demand was observed in the past:

Demand	14	15	16	17	18	19	20	21	22	23	24	25
Probability	3%	5%	7%	9%	11%	15%	18%	14%	8%	5%	3%	2%

LionAir does not book more than 19 passengers for each flight. Additional requests are always denied. However, there are often free seats despite a huge demand because of no-shows (passengers who book a flight but do not show up). Assume that the number of no-shows is described by a Poisson distribution with an expected value $\mu = 2$. Now, the airline decides to book 21 passengers for each flight in the future. Important costs and revenues are shown below:

- Every booked passenger that shows up and can be transported generates a revenue of 150 Euro. The revenue equals the profit because the variable costs are not significant.
- Every booked passenger that shows up and cannot be transported because more booked passengers than available seats show up generates costs of 325 Euro (refund of the ticket price + compensation payment).

a) Show the relationship between the input data and the resulting quantity in a deterministic model.

b) Which input data is stochastic and how is it distributed?

c) Perform a Monte Carlo simulation with 500 runs and answer the following questions:

   (1) What is the expecetd value and the standard deviation of the profit?

   (2) What is the probability that the profit is between 2,000 Euro and 2,500 Euro?

2 Linear programming

2.1 Modeling and Graphical Solutions

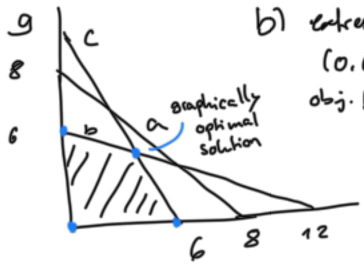
Question 43
Consider the linear program (LP) below:

Maximize

$$z = 40 \cdot x_1 + 30 \cdot x_2$$

b=c : $2x_1+x_2-12=2x_1+3x_2-18$

$\Rightarrow x_2=3 \Rightarrow x_1$ (in b) $= 4.5$



b) extreme points:

(0,0) ; (0,6) ; (4.5,3) ; (6,0)

obj. fun. values: just plug into function

subject to

$$
\begin{array}{lrcl}
a & 1 \cdot x_1 + 1 \cdot x_2 & \leq & 8 \\
b & 2 \cdot x_1 + 1 \cdot x_2 & \leq & 12 \\
c & 2 \cdot x_1 + 3 \cdot x_2 & \leq & 18 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

a) Illustrate the constraints in a chart and mark the feasible region.

b) Determine all extreme points of the feasible region and their objective function values.

c) Determine graphically an optimal solution(s) and specify the optimal value of the objective function.

Question 44
A farmer owns 100 ha. of land. He wants to plant potatoes in one part of the land and cereal crops in the other. Potatoes incur planting costs of 1,000 Euro per ha. and cereals 2,000 Euro per ha. Furthermore, it takes 1 workday to plant one ha. of potatoes and 4 workdays to plant one ha. of cereals. There are in total 160 workdays and 110,000 Euro available for the farmer. Expecting to make a profit of 1,000 Euro per ha. of potatoes and 3,000 Euro per ha. of cereals, he wants to determine how to maximize his total profits. # Potatoes $=x_1$; # Cereal $=x_2$

max $z=1000x_1+3000x_2$

s.t. $x_1+4x_2 \leq 160$

$1000x_1+2000x_2 \leq 110000$

$x_1+x_2 \leq 100$

$x_1 \geq 0; x_2 \geq 0$

a) Formulate the problem as a linear program.

b) Define and explain the objective function, the constraints and the decision variables.

c) Illustrate the constraints in a chart and mark the feasible region.

d) Determine graphically an optimal solution(s) and specify the optimal value of the objective function.

Question 45
A fish-processing factory produces pure crabmeat and crab salad. A crab picker needs 60 minutes to produce one kg of pure crabmeat. A cook takes 30 minutes to process one kg of pure crabmeat into one kg of crab salad. Two crab pickers work for the factory and work 5 hours a day. Furthermore, there is only one cook who works 3 hours a day. The profit which should be maximized is 3 Euro per kg of pure crabmeat and 5 Euro per kg of crab salad.

a) Formulate the problem as a linear program.

b) Define and explain the objective function, the constraints and the decision variables.

c) Illustrate the constraints in a chart and mark the feasible region.

d) Determine graphically an optimal solution(s) and specify the optimal value of the objective function.

Question 46
A mining company owns two different mines where certain kinds of ores are extracted. The mines are located in different areas and have different capacities. The ores can be divided in three different classes: Coarse-, medium- and fine-grained ore. Each class of ore has a certain demand. The mining company aims to provide a smelter a minimum of 12 tons of coarse-grained ore, 8 tons of medium-grained ore, and 24 tons of fine-grained ore per week. Mine 1 has operating costs of 200 Euro per day and Mine 2 has 160 Euro per day. In Mine 1, 6 tons of coarse-, 2 tons of medium- and 4 tons of fine-grained ore are extracted per day, while in Mine 2, 2 tons of coarse-, 2 tons of medium- and 12 tons of fine-grained ore are extracted per day.

a) Formulate the problem as a linear program to determine the minimum-cost extraction plan.

b) Define and explain the objective function, the constraints and the decision variables.

b) Illustrate the constraints in a chart and mark the feasible region.

c) Determine graphically an optimal solution(s) and specify the optimal value of the objective function.

Question 47
A brewery produces two different kinds of beer: Pilsener „Classic“ and the nonalcoholic „Free“. As an intern you should set a production plan to maximize the total revenue. There is no demand restriction for the two kinds of beer but the availability of hop and malt is limited. You carry a stock of 10,000 kg of malt and 2,000 kg of hop. To produce one litre of „Classic“ you need 200 g of malt and 9 g of hop. It can be sold for 1 Euro. To produce one litre of „Free“ you need 180 g of malt and 10 g of hop. It can be sold for 1.50 Euro. Furthermore there are advanced orders of 8,000 litres for „Classic“ which must be satisfied.

a) Formulate the problem as a linear program.

b) Define and explain the objective function, the constraints and the decision variables.

c) Illustrate the constraints in a chart and mark the feasible region.

d) Determine graphically an optimal solution(s) and specify the optimal value of the objective function.

e) How much should the price per litre of „Classic“ differ, so that it is optimal to produce 25,000 litres of „Classic“ and 25,000 litres of „Free“ ?

2.2 Simplex Algorithm: Basics

Question 48
Determine which conditions need to be satisfied to determine an optimal basic feasible solution of a linear program via simplex algorithm.

Question 49
Consider the linear program (LP) below:

Maximize

c)

$$z = 3 \cdot x_1 + 2 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$$

x_1 enters (largest coeff)

subject to

A

x_4 leaves (smallest ratio) 8/2=4

$$\begin{aligned} 1 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 &= 10 \\ 2 \cdot x_1 + \frac{3}{2} \cdot x_2 + 0 \cdot x_3 + \underline{1} \cdot x_4 + 0 \cdot x_5 &= 8 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 &= 4 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Variables in identity matrix are basic for initial basic solution

a) Determine the related basic feasible solution. $x_1, x_2 = 0$; $x_3 = 10$; $x_4 = 8$; $x_5 = 4$

b) Show that the basic feasible solution is feasible but not optimal. feasible: all constraints are ok ✓
not optimal: e.g. $x_1 = 1, \dots, x_3 = 9, x_4 = 6$ is better

c) Improve the basic feasible solution by performing one iteration of the simplex algorithm.

c) x_1 enters, x_4 leaves, 2 in constraint 2 is pivot

$0x_1 + \frac{1}{4}x_2 + 1x_3 - \frac{1}{2}x_4 + 0x_5 = 6$

$1x_1 + \frac{3}{4}x_2 + 0x_3 + \frac{1}{2}x_4 + 0x_5 = 4$

$0x_1 + 1x_2 + 0x_3 + 0x_4 + 1x_5 = 4$

$z - 12 = 0x_1 - \frac{1}{4}x_2 + 0x_3 - \frac{3}{2}x_4 + 0x_5$

Question 50
Consider the linear program (LP) below:

Maximize

$$6 \cdot x_1 - 12 \cdot x_2 - 4 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = z$$

subject to

$$\begin{aligned} 2 \cdot x_1 + 2 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 &= 5 \\ 1 \cdot x_1 + 4 \cdot x_2 + 4 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 &= 7 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

a) Determine the related basic feasible solution.

b) Show that the basic feasible solution is feasible but not optimal.

c) Improve the basic feasible solution by performing <u>one</u> iteration of the simplex algorithm.

Question 51
Consider the linear program (LP) below:

Maximize

$$3 \cdot x_1 + 1 \cdot x_2 + 4 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = z$$

subject to

$$\begin{aligned} 6 \cdot x_1 + 3 \cdot x_2 + 5 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 &= 25 \\ 3 \cdot x_1 + 4 \cdot x_2 + 5 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 &= 20 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

b	val	x_1	x_2	x_3	x_4	x_5
x_4	25	6	3	5	1	0
x_5	20	3	4	5^P	0	1
$-z$	0	3	1	4	0	0

a) Represent the linear program in a tableau format.

b) Show in the tableau that the basic feasible solution is feasible but not optimal. non-basic variables have obj. fun. coefficients > 0

c) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

① $\leftarrow x_3$ $\rightarrow x_5$

b	val	x_1	x_2	x_3	x_4	x_5
x_4	5	3^P	-1	0	1	-1
x_3	4	$3/5$	$4/5$	1	0	$1/5$
$-z$	-16	$3/5$	$-11/5$	0	0	$-4/5$

② $\leftarrow x_4$ $\rightarrow x_1$

b	val	x_1	x_2	x_3	x_4	x_5
x_1	$5/3$	1	$-1/3$	0	$1/3$	$-1/3$
x_3	3	0	1	1	$-1/5$	$2/5$
$-z$	-17	0	-2	0	$-1/5$	$-3/5$ ✓

→ no positive coefficients; optimal

Question 52
Consider the linear program (LP) below:

Maximize

$$1 \cdot x_1 + 1 \cdot x_2 + 3 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot x_7 = z$$

subject to

$$\begin{aligned} 1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot x_7 &= 10 \\ 1 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 + 0 \cdot x_6 + 0 \cdot x_7 &= 12 \\ -1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 + 0 \cdot x_7 &= 8 \\ 2 \cdot x_1 - 1 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 1 \cdot x_7 &= 2 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

a) Represent the linear program in a tableau format.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 53
(<u>Hint:</u> You should solve this question in less than 7 minutes.) √

Given a maximization problem and the simplex tableau below, answer the following questions.

BV	Wert	x_1	x_2	s_1	s_2	s_3
s_1	7	1	3	1		
s_2	2	1	2		1	
s_3	5	2	4			1
$-z$	-2	-3	$\boxed{1}$			

BV	val	x_1	x_2	s_1	s_2	s_3
s_1	4	$-1/2$	0	1	$-3/2$	0
x_2	1	$1/2$	1	0	$1/2$	0
s_3	1	0	0	0	-2	1
$-z$	-3	$-1/2$	0	0	$-1/2$	0

a) Which variable will leave the basis in the next pivot Step of the simplex algorithm? Justify your answer. s_2; x_2 will enter enter (only coefficient >0) and s_2 has the largest effect (smallest ratio)

b) Perform one pivot step of the simplex algorithm.

c) Do you find an optimal solution by performing the pivot Step in b)? If so, what is the optimal value? Justify your answer. Yes. The optimal value is -3. (at $x_1 = 0$; $x_2 = 1$; bc. x_1 is non-basic)

2.3 Transformation into Canonical Form

Question 54
Consider the linear program (LP) below:

Minimize

$$-1 \cdot x_1 - 2 \cdot x_2 = z$$

subject to

$$\begin{aligned} 1 \cdot x_1 + 1 \cdot x_2 &\leq 100 \\ 6 \cdot x_1 + 9 \cdot x_2 &\leq 720 \\ 0 \cdot x_1 + 1 \cdot x_2 &\leq 60 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Max.

$-z = 1x_1 + 2x_2$

s.t.

$$\begin{aligned} 1x_1 + 1x_2 + 1s_1 &= 100 \\ 6x_1 + 9x_2 + 1s_2 &= 720 \\ 1x_2 + 1s_3 &= 60 \end{aligned}$$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

BV	val	x_1	x_2	s_1	s_2	s_3
s_1	100	1	1	1		
s_2	720	6	9		1	
s_3	60		1 P			1
$-z$	0	1	2	0	0	0

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

1

BV	val	x_1	x_2	s_1	s_2	s_3
s_1	40	1		1		-1
s_2	180	6 P			1	-9
x_2	60		1			1
$-z$	-120	1	0	0	0	-2

2

BV	val	x_1	x_2	s_1	s_2	s_3
s_1	10			1		$1/2$
x_1	30	1			$1/6$	$-3/2$
x_2	60		1			1
$-z$	-150	0	0	0	$-1/6$	$-1/2$

optimal ✓

$z^* = 150$, $x_1^* = 30$; $x_2^* = 60$

Question 55
Consider the linear program (LP) below:

Maximize

$$5 \cdot x_1 - 7 \cdot x_2 + 3 \cdot x_3 = z$$

subject to

$$\begin{aligned} 1 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 &\leq 17 \\ 2 \cdot x_1 + 4 \cdot x_2 - 2 \cdot x_3 &\leq 11 \\ 1 \cdot x_1 + 1 \cdot x_2 - 3 \cdot x_3 &\geq -8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 56

Consider the linear program (LP) below:

Minimize

$$2 \cdot x_1 - 1 \cdot x_2 - 2 \cdot x_3 = z$$

max.

$-z = -2x_1 + 1x_2 + 2x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to

s.t.

$$\begin{aligned}
1 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 &\geq -9 \\
-1 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3 &= 0 \\
0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 &\leq 1 \\
x_1, x_2 &\geq 0 \\
x_3 &\in \mathbb{R}
\end{aligned}$$

$\cdot(-1)$

$-1x_1 - 2x_2 - 1x_3^+ + 1x_3^- + 1s_1 = 9$

$-1x_1 - 1x_3^+ + 1x_3^- + 1s_2 = 0$

$1x_1 + 1x_3^+ - 1x_3^- - s_3 = 0$

$1x_2 + 1x_3 + 1s_4 = 1$

$-x_1 - x_3 \geq 0 \to x_1 + x_3 \leq 0$

$-x_1 - x_3 \leq 0$

either Big-M, or if $= 0$, split into ≥ 0 and ≤ 0

$\searrow x_3^+ - x_3^-$

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 57

Consider the linear program (LP) of question 44.

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 58

Consider the linear program (LP) of question 45.

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

2.4 Dual Simplex Method / Big-M Method: Basics

Question 59

Determine which conditions need to be satisfied to determine an (optimal) basic feasible solution of a linear program via the dual simplex method. Must be dual-feasible → All coefficients in z must be ≤ 0

Question 60

Consider the linear program (LP) below:

Maximize

$$2 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + \overbrace{0 \cdot x_4}^{-Ma_1} + 0 \cdot x_5 = z$$

→ Not dual-feasible! Dual Simplex Method cannot be applied

subject to

$$\begin{aligned}
-2 \cdot x_1 + 5 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 &= 0 \\
1 \cdot x_1 - 2 \cdot x_2 + 0 \cdot x_3 - 1 \cdot x_4 + 0 \cdot x_5 &= -9 \\
2 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 &= 12 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

$\cdot(-1) \to$

violates identity matrix; add artificial variable to act as initial basic var.

$\to 1x_1 + 2x_2 + 0x_3 - 1x_4 + 1a_1 + 0x_5 = 9$

a)

BV	val	x_1	x_2	x_3	x_4	x_5	a_1
x_3	0	-2	5 P	1	0	0	0
a_1	9	1	2	0	-1	0	1
x_5	12	2	1	0	0	1	0
$-z$	0	2	1	0	0	0	$-M$
new $-z$	9M	M+2	2M+1	0	$-M$	0	0

← because a_1 is basic this must be 0! Transformed in new $-z$

Pivot (1)

BV	val	x_1	x_2	x_3	x_4	x_5	a_1
x_2	0	$-2/5$	1	$1/5$	0	0	0
a_1	9	$9/5$ P	0	$-2/5$	-1	0	1
x_5	12	$12/5$	0	$-1/5$	0	1	0
$-z$	9M	$\frac{9}{5}M + \frac{12}{5}$	0	$-\frac{2}{5}M - \frac{1}{5}$	$-M$	0	0

a) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

a) continue. Pivot 2

BV	val	x_1	x_2	x_3	x_4	x_5	~~a_1~~
x_2	2	0	1	1/9	-2/9	0	~~2/9~~
x_1	5	1	0	-2/9	-5/9	0	~~5/9~~
x_5	0	0	0	1/3	4/3	1	~~-4/3~~
$-z$	-12	0	0	1/3	4/3	0	~~-4/3-M~~

← M only in arbitrary coefficient; remove it

$(-\frac{2}{5}M-\frac{1}{5})-(\frac{9}{5}M+\frac{12}{5})\cdot(-\frac{2}{9})$

$-\frac{2}{5}M-\frac{1}{5}+\frac{2}{5}M+\frac{24}{45}=-\frac{9}{45}+\frac{24}{45}=\frac{15}{45}=\frac{1}{3}$

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

b) Pivot 3

BV	val	x_1	x_2	x_3	x_4	x_5
x_2	2	0	1		0	
x_1	5	1	0		0	
x_4	0	0	0	1/4	1	3/4
$-z$	-12	0	0	0	0	-1

→ optimal at $x_2^*=2; x_1^*=5; x_4^*=0$
$z^*=12$
(x_3, x_5 non-basic)

Question 61

Consider the linear program (LP) below:

Maximize

$$-1\cdot x_1+1\cdot x_2+0\cdot x_3+0\cdot x_4+0\cdot x_5 \quad = \quad z$$

subject to

$$\begin{aligned}
1\cdot x_1+3\cdot x_2+1\cdot x_3+0\cdot x_4+0\cdot x_5 &= 7\\
1\cdot x_1+\frac{1}{2}\cdot x_2+0\cdot x_3+1\cdot x_4+0\cdot x_5 &= -1\\
2\cdot x_1+2\cdot x_2+0\cdot x_3+0\cdot x_4+1\cdot x_5 &= 4\\
x_1,x_2,x_3,x_4,x_5 &\geq 0
\end{aligned}$$

a) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

b) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 62

Consider the linear program (LP) below:

Maximize

has to be 0 bc a_1 is basic

$$1\cdot x_1+2\cdot x_2 \underbrace{\quad}_{+0s_1-Ma_1} = \quad z$$

$z+2M=(M+1)x+(2-4M)x_2+0s_1+0a_1$

subject to

$3x_1-2x_2+s_1=10$

$$\begin{aligned}
3\cdot x_1-2\cdot x_2 &\leq 10\\
1\cdot x_1-4\cdot x_2 &= 2
\end{aligned}$$

$1x_1-4x_2+1a_1=2$

$$x_1,x_2,s_1,a_1 \geq 0$$

b)

BV	val	x_1	x_2	s_1	a_1
s_1	10	3	-2	1	0
a_1	2	1	-4	0	1
$-z$	2M	M+1	2-4M	0	0

1

BV	val	x_1	x_2	s_1	~~a_1~~
s_1	4	0	10	1	~~-3~~
x_1	2	1	-4	0	~~1~~
$-z$	-2	0	6	0	~~-M-1~~

c) 2

BV	val	x_1	x_2	s_1
x_2	0.4	0	1	0.1
x_1	3.6	1	0	0.4
$-z$	-4.4	0	0	-0.6

↳ optimal at $x_1^*=3.6; x_2^*=0.4$
$z^*=4.4$

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

c) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 63

Consider the linear program (LP) below:

Minimize ← Minimization! Here, all coefficients would need to be ≥0 for dual-feasibility.

$$-\frac{1}{2}\cdot x_1-\frac{1}{10}\cdot x_2+\frac{4}{7} \quad = \quad z$$

subject to

$$\begin{aligned}
x_1+x_2 &\leq 4\\
-x_1+3\cdot x_2 &\geq -6\\
x_1 &\geq 1\\
x_2 &\leq 1\\
x_1 &\geq 0\\
x_2 &\in \mathbb{R}
\end{aligned}$$

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

c) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 64
Consider the linear program (LP) below:

Minimize

$$3 \cdot x_1 - 2 \cdot x_2 + 1 \cdot x_3 - 4 \cdot x_4 = z$$

$\max -3x_1 + 2x_2 - 1x_3 + 4x_4 = -z$ → not dual-feasible

subject to

$$\begin{array}{rrcl}
-s_1 + a_1 & 1 \cdot x_1 + 1 \cdot x_2 - 4 \cdot x_3 + 2 \cdot x_4 & \geq & 4 \\
-s_2 & -3 \cdot x_1 + 1 \cdot x_2 - 2 \cdot x_3 + 0 \cdot x_4 & \leq & 6 \\
\cdot(-1) \mid \quad +a_2 & 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 - 1 \cdot x_4 & = & -1 \\
+a_3 & 1 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3 + 0 \cdot x_4 & = & 0 \\
x_1 = +x_1^+ - x_1^-;\ x_2 = +x_2^+ - x_2^- \leftarrow & x_1, x_2 & \in & \mathbb{R} \\
& x_3, x_4 & \geq & 0
\end{array}$$

$-z = -3x_1^+ + 3x_1^- + 2x_2^+ - 2x_2^- - x_3 + 4x_4 - Ma_1 - Ma_2 - Ma_3 \quad +M(1) + M(3) + M(4)$

$-z + 5M = (-3+2M)x_1^+ + (3-2M)x_1^- + (2+M)x_2^+ - (2+M)x_2^- - (1+5M)x_3 + (4+3M)x_4 - M \cdot s_1$

$x_1^+ - x_1^- + x_2^+ - x_2^- - 4x_3 + 2x_4 - s_1 + a_1 = 4 \quad (1)$

$-3x_1^+ + 3x_1^- + x_2^+ - x_2^- - 2x_3 + s_2 = 6 \quad (2)$

$-x_2^+ + x_2^- + x_4 + a_2 = 1 \quad (3)$

$x_1^+ - x_1^- + x_2^+ - x_2^- - x_3 + a_3 = 0 \quad (4)$

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

bro I'm not a computer, this will take way too long

c) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 65
Consider the linear program (LP) of question 46 without taking the constraints $x_1 \leq 7$ and $x_2 \leq 7$ into account.

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Find a basic feasible solution of the linear program by using the dual simplex method.

c) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 66
Consider the linear program (LP) of question 47.

a) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

b) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

c) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 67
A milk processing company produces cheese and cream. It takes 50 litres of milk to produce one kilogram of cheese and 4 litres of milk to produce 1 litre of cream. 1,500 litres of milk need to be processed every day. All products need to pass a strong quality control, which takes 5 minutes for each kilogram of cheese and 20 seconds for each litre of cream. An employee, who works 7.5 hours per day, carries out the quality control.

The maximum demands are 60 kilograms of cheese and 700 litres of cream per day. The cheese is sold for 5 Euro per kilogram and the cream for 2 Euro per litre.

The company aims at maximizing the daily revenues.

a) Formulate the problem as a linear program. Define and explain the objective function, the constraints and the decision variables.

b) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

c) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

d) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

Question 68

The owner of a fruit plantation has 80 tons of apples after the harvest. As he does not want to waste any apples he will either sell them for 0.50 Euro per kilogram or process them to fruit juice which can be sold for 1 Euro per litre. The fruit juice is a mixture of water and pure apple juice. One litre of pure apple juice can be obtained from 5 kilograms of apples. The water supply is unrestricted and for free.

The apples that are processed to apple juice need to be cleaned. The cleaning process takes 1.5 minutes per kilogram of apples. Moreover, it takes 1 minute to bottle 1 litre of fruit juice. Help laborers carry out these two tasks manually. The owner has a maximum of 2,500 working hours at his disposal. 20,000 litres of fruit juice need to be produced due to delivery commitments. The maximum demand of fruit juice is 70,000 litres and 60 tons of apples.

The owner aims at maximizing the sales revenue but he needs to obey a food law that sets a minimum proportion of pure apple juice to a.

a) Formulate the problem as a linear program. Define and explain the objective function, the constraints and the decision variables.

b) Is the given LP in canonical form? If not, transform it to a canonical form. Give an explanation.

c) Which values can a adopt so that

   (1) the linear program (LP) has no feasible solution?

   (2) the linear program (LP) is unbound?

   (3) all constraints that contain a are redundant?

d) Assume that a is equal to $10\% = 0.1$.

   (1) Find a basic feasible solution of the linear program by using the dual simplex method or the big-M method.

   (2) Perform pivot steps of the simplex algorithm in the tableau format until an optimal basic feasible solution is found.

a) $z = 0.5x_1 + 0.2x_2$
$x_1 + x_2 = 80000$
$1.7x_2 \leq 150000$
$0.2x_2 \geq 20000$
$0.2x_2 \leq 70000$
$x_1 \leq 60000$
$x_1, x_2 \geq 0$
$5x_2 \in \mathbb{N}$

2.5 Simplex Algorithm: Special Cases

Question 69

Describe the special cases that may appear in simplex algorithm.

Question 70

Consider the linear programs in tableau format:

(1)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	5			1	2	-1
x_2	4		1		1	0
x_1	2	1			-2	1
$-z$	-10				0	-1

Multiple Optimal Solutions: x_4 is non-basic but has coefficient 0

(2)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	2	-2	1	1		
x_4	0	5^P		-3	1	
x_5	7	2		1		1
$-z$	-6	3		-1		

Primal degeneracy: Basic leaving variable x_4 has value 0

(3)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	60	6	5	1		
x_4	80	6	5		1	
x_5	150	10	20			1
$-z$	0	500	450			

Redundant constraint: all solutions satisfying constraint 2 also satisfy constr. 1

(4)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	2	-2	-1	1		
x_4	4	1	-1		1	
x_5	1	0	-2			1
$-z$	-2	-1	3			

Unbound solution: All coefficients in the column for the pivot element are negative

(5)

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	60	6	5	1		
x_4	-80	6	5		1	
x_5	150	10	20			1
$-z$	0	500	450			

No feasible solution: Neither primal- (Simplex) nor dual (dual Simplex) feasible and Big-M artificial variable would not leave the basis (but how would I see that instantly?)

a) Specify the special case of each LP and describe how you can identify it in tableau format.

b) Specify in each case how these special cases affect the process of the simplex algorithm.

Question 71

Consider the linear program (LP) in tableau format:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_4	2		-4		1	1	3
x_3	1		0	1		2	3
x_1	α	1	1			0	-1
$-z$	10		0			-4	-5

How do you have to choose the parameter α so that the linear program (LP)

a) has at least one initial basic feasible solution? $\alpha \geq 0$

b) has at least one optimal solution which is not degenerate? $\alpha > 0$

c) has just one optimal solution which is degenerate? X

d) has several optimal solutions? $\alpha > 0$

e) has an unbounded feasible region?

Question 72

Consider the linear program (LP) in tableau format:

BV	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	1			0	1	-2
x_3	3			1	-1	3	-2
x_2	β		1		5	0	-1
$-z$	-20				-2	-4	β

How do you have to choose the parameter β so that the linear program (LP)

a) has at least one initial basic feasible solution?

b) has at least one optimal solution which is not degenerate?

c) has just one optimal solution which is degenerate?

d) has several optimal solutions?

e) has an unbounded feasible region?

2.6 Sensitivity Analysis: Shadow Price and Reduced Cost

Question 73
Consider the optimal tableau of question 57.

a) Determine and interpret the shadow prices of the three production factors.

b) The farmer considers growing another product. Which product should the farmer in your opinion choose if he can select between the following products?

Production Factors	Products: Asparagus	Products: Corn
Plantable area	3	3
Working Days	4	2
Budget	2,000	2,000
Profit Margin	4,000	2.500

c) The Farmer wife suggests to grow both products. Do you have the same opinion? Why and Why not?

Question 74
Consider a company that produces two kinds of pralines, which are made of chocolate and nougat. The available capacities, the needed amount of chocolate and nougat for the production of the products „Chocolate Dream“ and „Nougat Hill“ and their contribution margins are listed in the following table:

Production Factors	Products: Chocolate Dream (x_1)	Products: Nougat Hill (x_2)	Capacity
Chocolate (x_3)	3	1	20
Nougat (x_4)	1	2	15
Contribution Margin	4	3	

A company's employee already found an optimal tableau by using the simplex algorithm:

BV	Value	x_1	x_2	x_3	x_4
x_1	5	1		$\frac{4}{10}$	$-\frac{2}{10}$
x_2	5		1	$-\frac{2}{10}$	$\frac{6}{10}$
$-z$	-35			-1	-1

a) How much should the company be willing to pay for one additional unit of chocolate capacity at the most? 1 MU

b) A praline of the kind „Calorie Bomb“ is made of 4 units of chocolate and 3 units of nougat. What is the contribution margin of the „Calorie Bomb“ so that the company would produce it?

Using the respective shadow prices: 4(1)+3(1)=7

2.7 Sensitivity Analysis: Variation of the objective function coefficient

Question 75

Consider a brewery which produces three kinds of beer with hop and malt. You find the given capacities, the required amount and their contribution margin in the table below.

Production Factors	Products			Capacity
	Wheat Beer (x_1)	Pils (x_2)	Export (x_3)	
Hop (x_4)	1	1	2	7
Malt (x_5)	2	1	3	10
Contribution Margin	8	6	5	

An employee has already determined an optimal tableau of the simplex algorithm:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_2	4		1	1	2	-1
x_1	3	1		1	-1	1
$-z$	-48			-9	-4	-2

a) What is the contribution margin of „Export“ so that it is in the basis? Using reduced cost: 5+9 = 14

b) How much can the contribution margin of „Wheat Beer“ change without changing the optimal basis? Add Δ to C.M., -> apply to optimized (0+Δ) -> Set to 0 bc. basic var., find limit in other coeff.

Question 76

Consider the linear program (LP):

Maximize

$$-1 \cdot x_1 + 1 \cdot x_2 = z$$

subject to

$$\begin{aligned} 1 \cdot x_1 + 3 \cdot x_2 &\leq 7 \\ 1 \cdot x_1 + 2 \cdot x_2 &\geq 1 \\ 2 \cdot x_1 + 2 \cdot x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

and the corresponding optimal tableau:

BV	Value	x_1	x_2	x_3	x_4	x_5
x_3	1	-2	0	1	0	$-\frac{3}{2}$
x_2	2	1	1	0	0	$\frac{1}{2}$
x_4	3	1	0	0	1	1
$-z$	-2	-2	0	0	0	$-\frac{1}{2}$

a) In which interval can the objective function coefficient of the variable x_1 change without changing the optimal basis? shadow price = 2 $\Rightarrow \Delta \leq 2$

b) In which interval can the objective function coefficient of the variable x_2 change without changing the optimal basis? $-2-\Delta \leq 0 \wedge -\frac{1}{2}-\frac{1}{2}\Delta \leq 0 \Rightarrow -1 \leq \Delta$

c) Consider now the following extended linear program. Determine the parameter α so that the variable x_3 is in the optimal basis. Pricing Out

Maximize

$$-1 \cdot x_1 + 1 \cdot x_2 + \alpha \cdot x_3 = z$$

subject to

$$\begin{aligned}
1 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 &\leq 7 \\
1 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 &\geq 1 \\
2 \cdot x_1 + 2 \cdot x_2 + 5 \cdot x_3 &\leq 4 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

Question 77

Daddy Cool sells two kinds of punch at the Munich Christmas Market: Cinnamon and Clove. One litre cinnamon punch includes 0.7 litres of red wine, 0.25 litres of water, juice of 1.5 lemons and 3 grams of cinnamon. One litre Clove punch includes 0.8 litres of red wine, 0.15 litres of water, juice of 0.5 lemons and 5 grams of clove. There are in total 35 litres of red wine, 10 litres of water, 40 lemons, 75 gram cinnamon and 200 gram clove available for Daddy Cool. It takes 2.7 minutes to measure and mix the components for one litre of Cinnamon punch and 1.8 minutes for one litre of Clove punch. Due to the fact that he sells and serves the punch on his own, he got at most 90 minutes to mix the components. The contribution margin of one litre of Cinnamon punch is 0.7 Euro and 0.4 Euro per litre of Clove punch. Assume that all of the produced punch can be sold. How many litres of Cinnamon punch and Clove punch should Daddy Cool produce to maximize his contribution return?

a) Formulate the problem as a linear program and explain the objective function, the constraints and the decision variables.

b) Determine the optimal solution and the optimal objective function value with the simplex algorithm.

c) How much does the optimal objective function value differ if there would be one lemon less?

d) A friend offers to measure and mix the components. How high should his maximum hourly salary be and how long should he work for Daddy Cool at the most?*

e) He plans to offer another punch out of 0.75 litres of red wine, 0.2 litres of water and the juice of one lemon and a contribution return of 0.30 Euro per litre. It takes 1 minute to measure and mix the components. Should he expand his range of products with that punch?

f) Determine the interval, in which the contribution return of one litre of Cinnamon punch can change without changing the optimal basis.

g) Assume that unused lemons could be sold to a chummy fruit seller. Specify the contribution return per lemon so that the optimal basis will change.

* The method to solve the second part of this question is applicable to the Master level only.

2.8 Dual Programs: Basics

Question 78

Explain why it can be practical to solve the relating dual instead of the linear program.

Question 79

Determine the relating dual to the following linear program:

Maximize

$$3 \cdot x_1 + 2 \cdot x_2 = z$$

Primal

max $3x_1 + 2x_2 = z$

subject to

$$
\begin{aligned}
1 \cdot x_1 + 2 \cdot x_2 &\leq 10 \\
2 \cdot x_1 + \frac{3}{2} \cdot x_2 &\leq 8 \\
0 \cdot x_1 + 1 \cdot x_2 &\geq 4 \quad \rightarrow y_3 \leq 0 \\
x_1 &\geq 0 \\
x_2 &\leq 0
\end{aligned}
$$

$\rightarrow$ flip 2nd constraint

Dual

min $v = 10y_1 + 8y_2 + 4y_3$

s.t.

$1y_1 + 2y_2 \geq 3$

$2y_1 + \frac{3}{2}y_2 + 1y_3 \leq 2$

$y_1, y_2 \geq 0$; $y_3 \leq 0$

Question 80
Determine the relating dual to the following linear program:

Maximize

$$2 \cdot x_1 + 1 \cdot x_2 = z$$

subject to

$$
\begin{aligned}
y_1 \quad & 2 \cdot x_1 - 5 \cdot x_2 = 0 \quad \rightarrow y_1 \text{ unbounded} \\
y_2 \quad & 1 \cdot x_1 + 2 \cdot x_2 \geq 9 \quad \rightarrow y_2 \leq 0 \\
y_3 \quad & 2 \cdot x_1 + 1 \cdot x_2 \leq 12 \\
& x_1 \geq 0 \\
& x_2 \in \mathbb{R}
\end{aligned}
$$

$\rightarrow$ 2nd constraint =

Dual

min $v = 0y_1 + 9y_2 + 12y_3$

s.t.

$2y_1 + 1y_2 + 2y_3 \geq 2$

$-5y_1 + 2y_2 + 1y_3 = 1$

$y_1 \in \mathbb{R}$; $y_2 \leq 0$; $y_3 \geq 0$

2.9 Dual Programs: Properties

Question 81
Define and explain the four duality properties.

Question 82
Consider the linear program of question 44 again.

Primal

max $z = 1000x_1 + 3000x_2$

s.t.

$x_1 + x_2 \leq 100$

$x_1 + 4x_2 \leq 160$

$1000x_1 + 2000x_2 \leq 110000$

$x_1, x_2 \geq 0$

Dual

min $v = 100y_1 + 160y_2 + 110000y_3$

s.t.

$y_1 + y_2 + 1000y_3 \geq 1000$

$y_1 + 4y_2 + 2000y_3 \geq 3000$

$y_1, y_2, y_3 \geq 0$

a) Determine the relating dual linear program.

b) Determine the optimal solution of the dual linear program.

c) In which range are the objective function values of all basic feasible solutions of the dual linear program located?

Question 83
Reconsider the linear program of question 46 but without taking the constraints $x_1 \leq 7$ and $x_2 \leq 7$ into account.

a) Transform the LP into the relating dual linear program.

b) Solve the dual linear program with the simplex algorithm.

c) Compare the optimal tableau format from b) with the optimal tableau format in question 23. What do you notice? Which duality properties are shown?

3 Integer and Mixed-Integer Programming

3.1 Branch-and-Bound: Basics

Question 84
Explain the meaning of an LP-relaxation of an integer or a mixed-integer program and how its optimal solution is related to that of the linear program.

Question 85
Describe the general procedure and the components of the branch-and-bound method.

3.2 Branch-and-Bound with Simplex Algorithm

Question 86
Consider the integer program below:

Maximize

$$z = 3 \cdot x_1 + 4 \cdot x_2$$

subject to

$$\begin{aligned} 6 \cdot x_1 + 5 \cdot x_2 &\leq 25 \\ 3 \cdot x_1 + 5 \cdot x_2 &\leq 20 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the MUB rule and the smallest number rule as tie-breaker. Stop the branch-and-bound procedure when the objective function value of your solution is at most 5% smaller than the optimal objective function value.

The following information is available to solve the LP-relaxation:

① branch into P_b, P_e → Next node P_b (higher UB)

Subproblem	To Initial Problem P_0 Added Constraints		Solution of the Relaxation x_1	x_2	z
P_a ✓			$\frac{5}{3}$	3	17
P_b ✓	$x_1 \leq 1$		1	3.4	16.6
P_c ✓	$x_1 \leq 1$	$x_2 \geq 4$	0	4	16
P_d O	$x_1 \leq 1$	$x_2 \leq 3$	1	3	15
P_e ✓	$x_1 \geq 2$		2	2.6	16.4
P_f O	$x_1 \geq 2$	$x_2 \leq 2$	2.5	2	15.5
P_g O	$x_1 \geq 2$	$x_2 \geq 3$	infeasible		
P_h	$x_1 = 2$	$x_2 \leq 2$	2	2	14
P_i	$x_1 \geq 3$	$x_2 \leq 2$	3	1.4	14.6
P_j	$x_1 \geq 3$	$x_2 = 2$	infeasible		
P_k	$x_1 \geq 3$	$x_2 \leq 1$	3	1	13

② branch → P_c, P_d → Next P_e (highest UB in list of active nodes)

④ Integer ✓ → Check remaining nodes' bounds

③ branch → P_f, P_g → Next P_c

⑤ Active nodes P_d, P_f, P_g all have lower bounds than integer solution P_c → Solution found

Question 87
Consider the integer program below:

Minimize

$$z = 3 \cdot x_1 + \frac{5}{2} \cdot x_2$$

subject to

$$\begin{aligned} 2 \cdot x_1 + 3 \cdot x_2 &\geq 19 \\ 3 \cdot x_1 + \frac{12}{7} \cdot x_2 &\geq 22 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

a) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the MLB rule and the smallest number rule as tie-breaker. Solution is $x_1 = 6$, $x_2 = 3$, $z = 25.5$

b) When could you have stopped the branch-and-bound procedure given that the objective function value of your solution is at most 8% higher than the optimal objective function value? When $z \leq 27.54$
However, the first integer solution found is also the optimal solution → No early termination

The following information is available to solve the LP-relaxation:

Subproblem	To Inital Problem P_0 Added Constraints		Solution of the Relaxation x_1	x_2	z	
P_a			6	2.33	23.83	① Branch $P_b, P_e \to P_b$
P_b		$x_2 \geq 3$	5.6	3	24.35	② Branch $P_c, P_d \to P_e$
P_c	$x_1 \leq 5$	$x_2 \geq 3$	5	4.08	25.21	⑤ Branch $P_j, P_k \to P_d$
P_d	$x_1 \geq 6$	$x_2 \geq 3$	6	3	25.5	⑥ Feasible & better than remaining active nodes
P_e		$x_2 \leq 2$	6.5	2	24.5	③ Branch $P_f, P_g \to P_g$
P_f	$x_1 \leq 6$	$x_2 \leq 2$	infeasible			
P_g	$x_1 \geq 7$	$x_2 \leq 2$	7	1.67	25.17	④ Branch $P_h, P_i \to P_c$
P_h	$x_1 \geq 7$	$x_2 = 2$	7	2	26	
P_i	$x_1 \geq 7$	$x_2 \leq 1$	8	1	26.5	
P_j	$x_1 \leq 5$	$x_2 \leq 4$, $x_2 \geq 3$	infeasible			
P_k	$x_1 \leq 3$	$x_2 \geq 5$	3	7.58	27.95	
P_l	$x_1 \leq 5$	$x_2 \geq 5$	4.47	5	25.91	
P_m	$x_1 = 5$	$x_2 \leq 4$	Infeasible			
P_n	$x_1 \leq 4$	$x_2 \geq 5$	4	5.83	26.58	

3.3 Branch-and-Bound without Simplex Algorithm

Question 88
A student considers buying a ring, a necklace and a bracelet for a party on Friday evening. But she has only 10 Euro.The necklace costs 7 Euro, the ring 2 Euro and the bracelet 4 Euro. She likes the necklace more than the bracelet, so it has more value to her than the bracelet. She also likes the bracelet more than the ring. The costs and the values are:

Jewelery	Necklace	Ring	Bracelet
Value	4	2	3
Cost	7	2	4

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the LIFO rule and the smallest number rule as tie-breaker. (The student can buy at most one piece of each jewelery type.)

Question 89
A student is searching for a job and has three opportunities. The first employer is a chair at a university and pays 60 Euro per week for 8 hours work. The second employer is Sieh&Menz and pays 80 Euro per week for 8 hours work. The third employer is McConsult and pays 90 Euro per week for 6 hours work. The students wants to earn as much money as possible but he has only 20 hours per week available.

Employer	Chair	Sieh&Menz	McConsult
Salary	60	80	90
Hours	8	8	6

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the MUB rule and the biggest number rule as tie-breaker.

Question 90
You decided to prepare yourself without interruption for the exam „Management Science“ on a lonely island. The instruction of the airline states that the maximum total weight of your luggage should not exceed 12kg. Firstly you list all objects with the information of their weights and their subjective benefits:

Object	1	2	3	4	5	6
Weight in kg	1	2	2	4	6	10
Benefit	8	8	6	10	12	12

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the LIFO rule and the smallest number rule as tie-breaker.

Question 91
An express agent transports orders from Munich to Hamburg. For the next ride he has a truck with the maximum capacity of 10 tons. Six orders are available:

Order	1	2	3	4	5	6
Return in Euro	9,000	4,000	3,000	11,000	3,500	12,000
Weight in Tons	5	3	2	6	2	7

Which order should the driver pick for his next ride, if he wants to maximize the total return of this tour?

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the MUB rule and the smallest number rule as tie-breaker.

Question 92
OZON AG wants to decrease their pollutant emission by at least 7 units. They have to choose between 3 different decreasing alternatives A,B and C. The following table presents the cost of each alternative and by how much emission the alternative will decrease. The target of OZON AG is to decrease their pollutant emission at minimum cost.

Alternative	A	B	C
Costs	7	4	6
Decrease	5	3	4

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the MLB rule and the smallest number rule as tie-breaker. Stop the branch-and-bound procedure when the objective function value of your solution is at most 4% higher than the optimal objective function value.

Question 93

The company plans to build one or two furniture warehouses to supply its three stores, aiming to minimize the sum of warehouse- and transportation costs. The following table presents the fixed costs to build warehouses:

Warehouse	1	2
Fixed costs	85	70

Each store has the following demand:

Store	1	2	3
Demand	11	14	9

The following table presents the transportation cost per unit shipped from each warehouse to each store.

	Store		
Warehouse	1	2	3
1	8	5	6
2	7	9	8

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution by the branch-and-bound approach. Consider the subproblems using the LIFO rule and the biggest number rule as tie-breaker.

Question 94

A company wants to build at most three warehouses to supply its six stores. The following table presents the fixed costs to build warehouses:

Warehouse	1	2	3
Fixed costs	97	74	83

Each store has the following demand:

Store	1	2	3	4	5	6
Demand	10	17	21	13	24	19

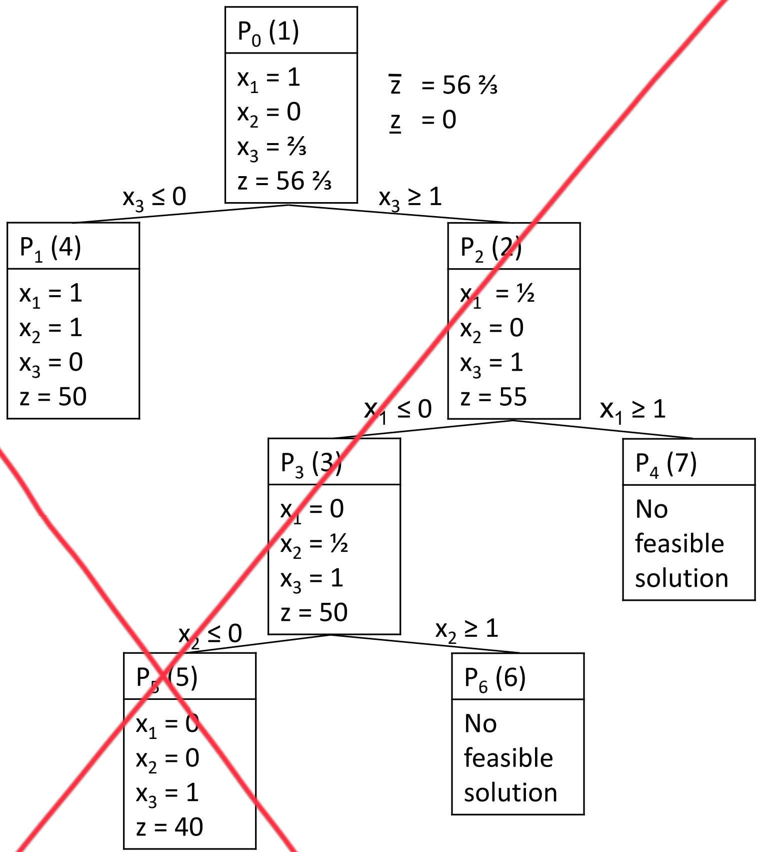
The following table presents the transportation cost per unit shipped from each warehouse to each store.

	Store					
Warehouse	1	2	3	4	5	6
1	17	9	10	9	11	6
2	10	12	6	7	8	10
3	9	6	7	13	14	8

a) Determine good upper and lower bounds for the objective function value of an optimal solution.

b) Determine an optimal solution with branch-and-bound. Consider the subproblems using to the LIFO rule and the smallest number as tie-breaker. Stop the branch-and-bound procedure when the objective function value of your solution is at most 25% higher than the optimal objective function value.

Question 95
Consider the following branch-and-bound-Tree and answer the subsequent questions:



a) Is this a maximization or minimization problem?

b) Which rule and tie-breaker were used to choose subproblems?

c) Why was the variable x_2 branched after solving subproblem P_3?

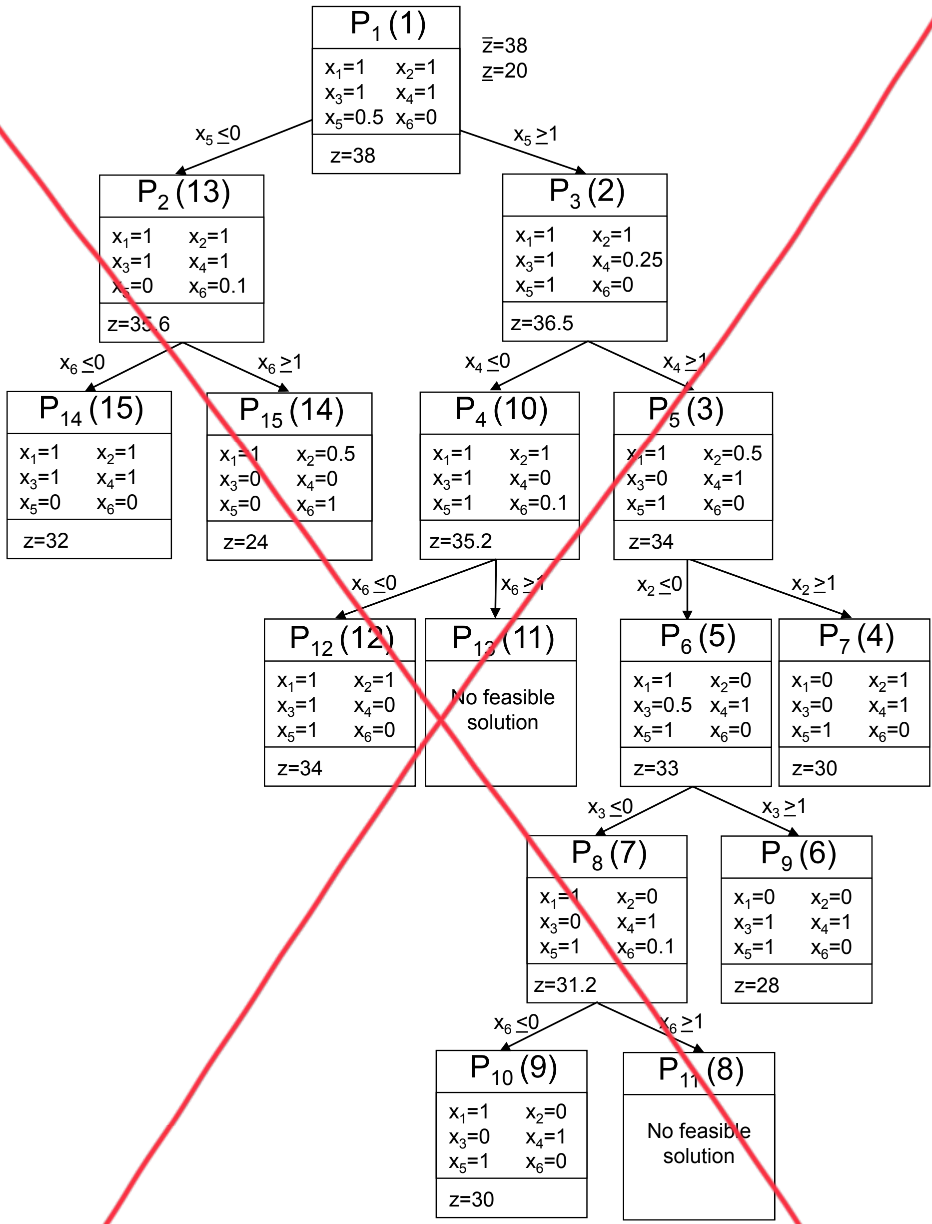
d) Explain for each subproblem why it was not branched and, where necessary, when the lower bound could be updated.

e) What is an optimal solution and the optimal objective function value of this problem?

f) When and with which solution would you end the branch-and-bound procedure, given that a maximum deviation of 15% would be allowed?

g) Derive a related binary integer program (BIP) out of the branch-and-bound tree.

Question 96
Consider the following branch-and-bound-Tree and answer the subsequent questions:



a) Is this a maximization or minimization problem?

b) Which rule and tie-breaker were used to choose subproblems?

c) Why was the variable x_3 branched after solving subproblem P_6?

d) Explain for each subproblem why it was not branched and, where necessary, when the lower bound could be updated.

e) What is an optimal solution and the optimal objective function value of this problem?

f) When and with which solution would you end the branch-and-bound procedure, given that a maximum deviation of 20% would be allowed?

4 Graph Theory and Network Flow Models

4.1 Shortest Path Problems: Dijkstra's Algorithm

Question 97
Determine which prerequisites need to be satisfied to determine the shortest path from an initial node to all nodes of a network by using Dijkstra's algorithm.

Question 98
How many iterations of Dijkstra's algorithm are needed to determine the shortest path from any initial node to every other node in a network, given a graph defined by a set of n nodes?

Question 99

init

i	1	2	3	4	5	6
d	0	∞	∞	∞	∞	∞
p						

M: 1

1

i	1	2	3	4	5	6
d	0	5	4	∞	∞	∞
p		1	1			

M: 2, 3

2

i	1	2	3	4	5	6
d	0	5	4	∞	7	∞
p		1	1		3	

M: 2, 5

3

i	1	2	3	4	5	6
d	0	5	4	11	7	∞
p		1	1	2	3	

M: 5, 4

4

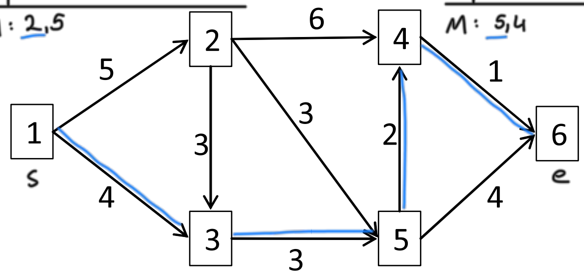
i	1	2	3	4	5	6
d	0	5	4	9	7	11
p		1	1	5	3	5

M: 4, 6

4

i	1	2	3	4	5	6
d	0	5	4	9	7	10
p		1	1	5	3	4

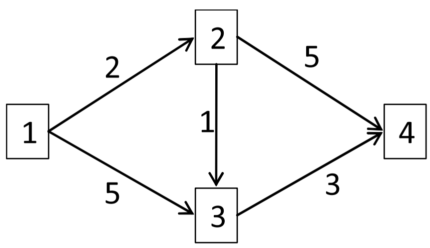
M: 6 → No arcs from 6



Shortest path: 6 ← 4 ← 5 ← 3 ← 1
or 1 → 3 → 5 → 4 → 6

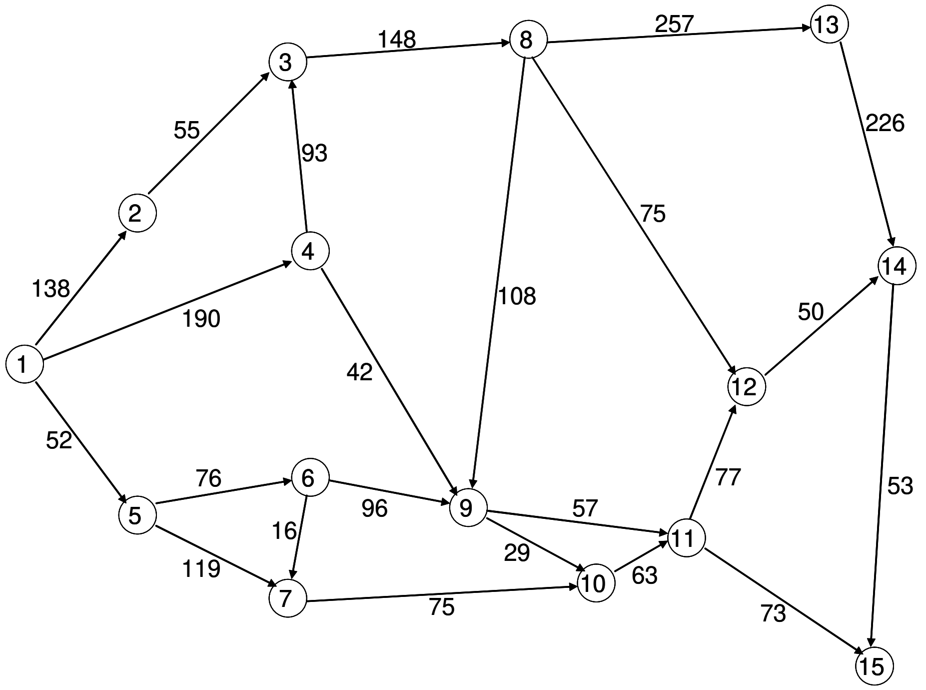
Determine the shortest path from node 1 to node 6 by using Dijkstra's algorithm.

Question 100



Determine the shortest path from node 1 to node 4 by using Dijkstra's algorithm.

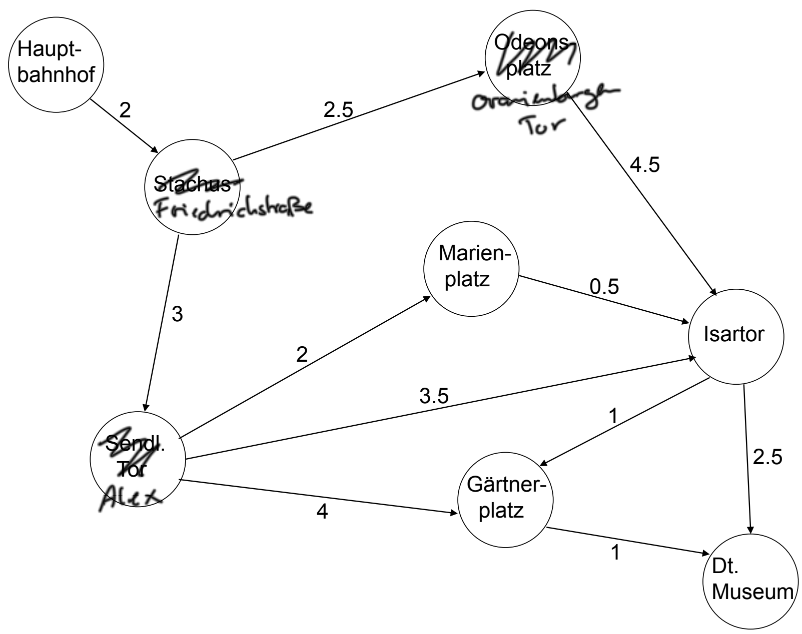
Question 101
Consider the graph below:



a) Determine the shortest path from node 1 to node 3 by using Dijkstra's algorithm.

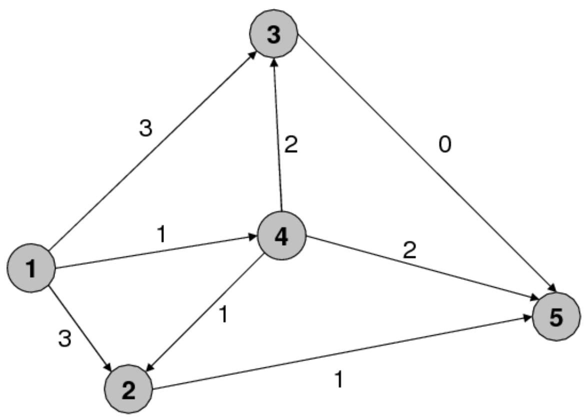
b) At which point could you stop the algorithm when you only want to determine the shortest path to node 7?

Question 102
A taxi driver is supposed to take a passenger, who is not familiar with the city, from "Hauptbahnhof" to "Dt. Museum". A map of Munich's city center that shows the most important places and journey times is presented in a (simplified) graph:



Determine the shortest path from "Hauptbahnhof" to "Dt. Museum" by using Dijkstra's algorithm.

Question 103
One of your fellow students wants to find the shortest paths from node 1 to all other nodes in the network by using Dijkstra's algorithm:



a) He has already performed the first iteration of the algorithm (see in the table below). Perform the second iteration of the algorithm.

i	1	2	3	4	5
$D[i]$	0	3	3	1	∞
$R[i]$		1	1	1	
$M = \{2, 3, 4\}$					

2

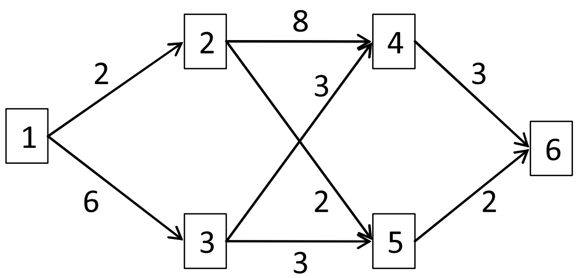
i	1	2	3	4	5
d	0	2	3	1	3
p		4	1	1	4
M	2, 3, 5				

b) Which shortest paths are known after the third iteration of the algorithm?

From node 1 to nodes 2, 4 (as they were removed from M)

4.2 Shortest Path Problems: Floyd-Warshall Algorithm

Question 104



init	1	2	3	4	5	6
1	0;1	2;1	6;1			
2		0;2		8;2	2;2	
3			0;3	3;3	3;3	
4				0;4		3;4
5					0;5	2;5
6						0;6

k=1 has no i

k=2	1	2	3	4	5	6
1	0;1	2;1	6;1	10;2	4;2	
2		0;2		8;2	2;2	
3			0;3	3;3	3;3	
4				0;4		3;4
5					0;5	2;5
6						0;6

k=3	1	2	3	4	5	6
1	0;1	2;1	6;1	9;3	4;2	
2		0;2		8;2	2;2	
3			0;3	3;3	3;3	
4				0;4		3;4
5					0;5	2;5
6						0;6

j={4,5}

k=4	1	2	3	4	5	6
1	0;1	2;1	6;1	9;3	4;2	12;4
2		0;2		8;2	2;2	11;4
3			0;3	3;3	3;3	6;4
4				0;4		3;4
5					0;5	2;5
6						0;6

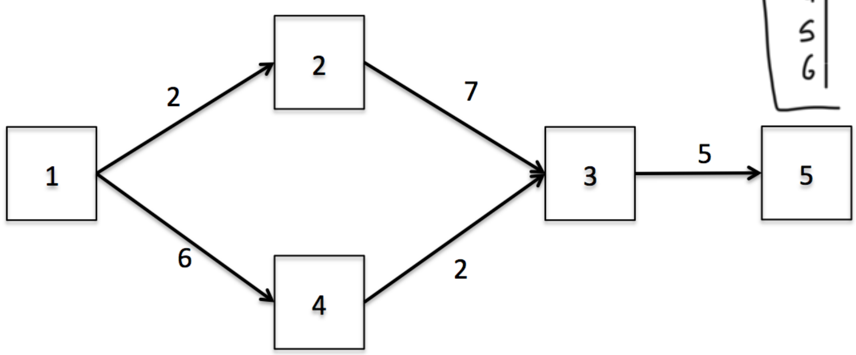
j={6}

Determine the shortest path from node 1 to node 6 by using the Floyd-Warshall algorithm. State the shortest path and its length.

k=5	1	2	3	4	5	6
1	0;1	2;1	6;1	9;3	4;2	6;5
2		0;2		8;2	2;2	4;5
3			0;3	3;3	3;3	5;5
4				0;4		3;4
5					0;5	2;5
6						0;6

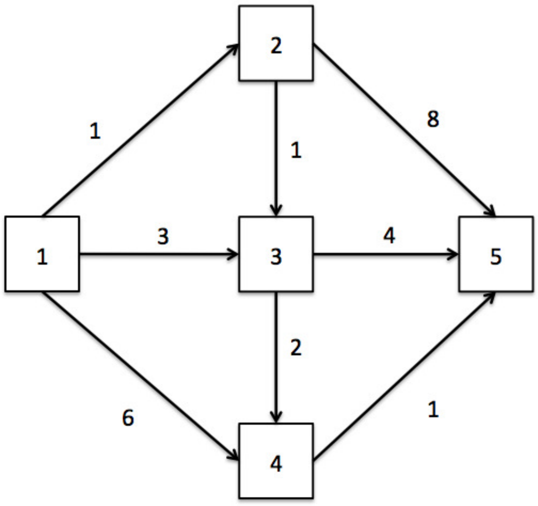
j={6}

Question 105



Determine the shortest path from node 1 to node 5 by using the Floyd-Warshall algorithm. State the shortest path and its length.

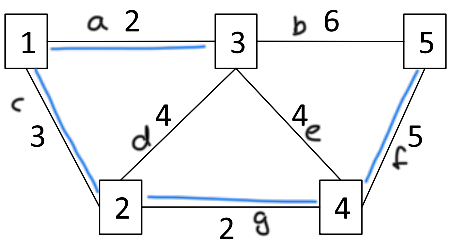
Question 106

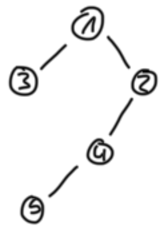


Determine the shortest path from node 1 to node 5 by using the Floyd-Warshall algorithm. State the shortest path and its length.

4.3 Minimum Spanning Tree Problems: Kruskal's Algorithm

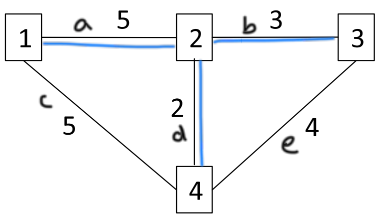
Question 107
Consider the graph below: Sorted edges: a, g, c, d, e, f, b stopped at edges=n-1





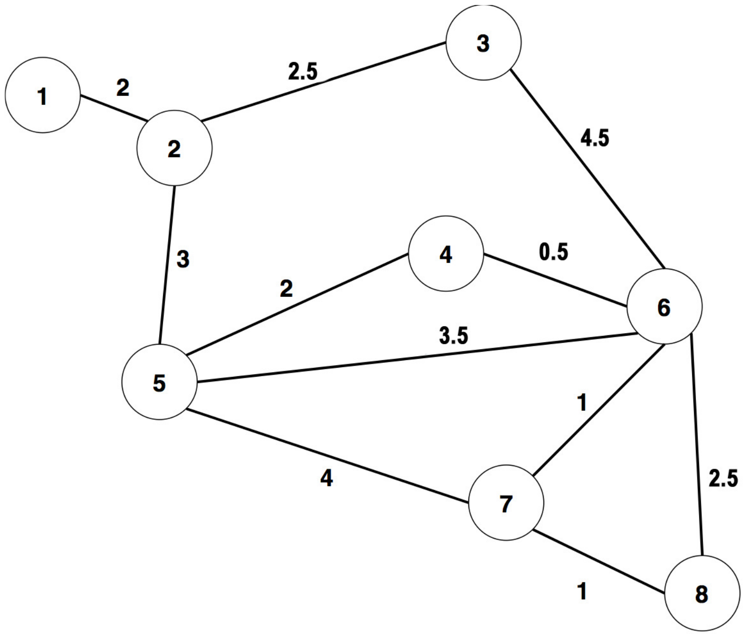
Determine a minimum spanning tree by using Kruskal's algorithm. Stop the algorithm as soon as possible.

Question 108
Consider the graph below: L= d, b, e, a, c stop



Determine a minimum spanning tree by using Kruskal's algorithm. Stop the algorithm as soon as possible.

Question 109
Consider the graph below:



Determine a minimum spanning tree by using Kruskal's algorithm. Stop the algorithm as soon as possible.

Question 110
Consider the graph of question 101 again and assume that all arcs are replaced by (undirected) edges.

a) Determine a minimum spanning tree by using Kruskal's algorithm.

b) Explain why you can stop Kruskal's algorithm after you have added 14 arcs, even though there are still arcs in list L. 15 nodes -> tree has n-1 edges

4.4 Transportation Problem: Northwest Corner Rule

Question 111
Consider a classic transportation problem with three suppliers and four customers. The cost matrix, the supplies and the demands are listed below:

$$c = \begin{pmatrix} 2 & 3 & 11 & 7 \\ 1 & 0 & 6 & 1 \\ 5 & 8 & 15 & 9 \end{pmatrix}$$

$a = \begin{pmatrix} 6 & 1 & 10 \end{pmatrix}$ $\qquad b = \begin{pmatrix} 7 & 5 & 3 & 2 \end{pmatrix}$

Find a feasible solution by using the northwest corner rule.

Question 112
Consider a classic transportation problem with three suppliers and four customers. The cost matrix, the supplies and the demands are listed below:

$$c = \begin{pmatrix} 4 & 5 & 2 & 10 \\ 5 & 8 & 10 & 2 \\ 10 & 10 & 4 & 3 \end{pmatrix}$$

$a = \begin{pmatrix} 5 & 8 & 7 \end{pmatrix}$ $\qquad b = \begin{pmatrix} 3 & 6 & 6 & 5 \end{pmatrix}$

Find a feasible solution by using the northwest corner rule.

Question 113
A company operates distilleries in four different cities and wheat is supplied by three agricultural cooperatives. The distilleries' demands of wheat in tons are:

$b = \begin{pmatrix} 30 & 40 & 20 & 10 \end{pmatrix}$

The maximum supplies of wheat in tons by the cooperatives are

$a = \begin{pmatrix} 100 & 50 & 100 \end{pmatrix}$

The transportation cost for one ton of wheat from cooperative i to distillery j is given in the following matrix:

$$c = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 2 & 4 & 3 & 2 \\ 4 & 3 & 3 & 4 \end{pmatrix}$$

Find a feasible solution by using the northwest corner rule.

4.5 Allocation Problem: Northwest Corner Rule

Question 114
Consider the following cost matrix:

$$c = \begin{pmatrix} 4 & 5 & 7 & 3 & 6 \\ 1 & 3 & 5 & 8 & 4 \\ 0 & 6 & 5 & 7 & 2 \\ 3 & 5 & 6 & 3 & 6 \\ 9 & 3 & 4 & 3 & 4 \end{pmatrix}$$

Find a feasible solution by using the northwest corner rule.

Question 115
The young married couple, Stefan and Sasha, wants to share the most important domestic work (shopping, cooking, laundry and changing the baby's diaper). Each is supposed to take care of exactly two tasks, so that the overall time spent is minimized. Both are willing to handle any task without complaining. Though, they do not perform the different tasks at the same efficiency. The following table lists the time spent on each task by each person.

Weekly hours for	Shopping	Cooking	Laundry	Changing diaper
Sasha	3.6	7.8	2.9	4.5
Stefan	4.3	7.2	3.1	4.9

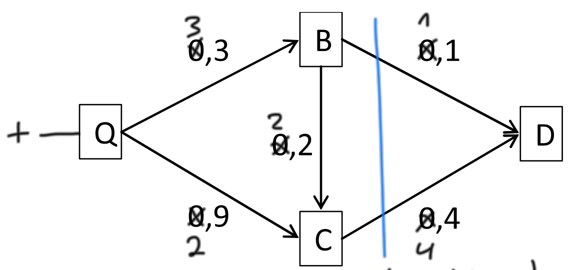
Find a feasible solution by using the northwest corner rule.

4.6 Maximum flow problems: Ford-Fulkerson algorithm

No determining of feasible start flows

Question 116
Determine which prerequisites need to be satisfied to find the maximum flow of a graph via the Ford-Fulkerson algorithm.

Question 117
The following network with arc capacities, source Q and sink D is given.



Cut here bc only D wasn't labelled in final iteration

a) Modify the graph so that you are able to determine its maximum flow by using the Ford-Fulkerson algorithm.

	1	2	3	4
B	Q+3	Q+2	—	C-2
C	—	B+2	Q+9	Q+7
D	B+1	C+2	C+2	—
Δ	1	2	2	0
Σ	1	3	5	5

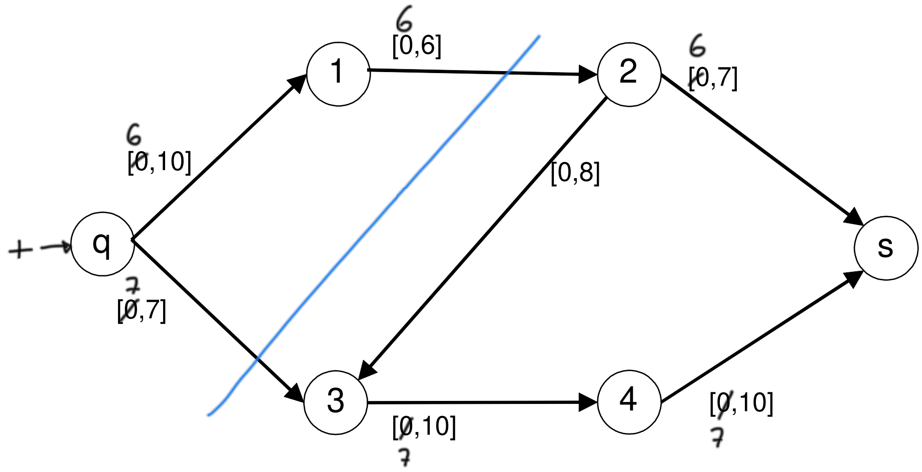
b) Determine a feasible (not necessarily optimal) flow from the source to the sink. Start out from that flow and perform as many iterations of the Ford-Fulkerson algorithm as necessary to determine the maximum flow from the source to the sink.

c) Determine the cut with the minimum capacity. Mark the cut in the graph and determine its capacity.

Question 118

The following network with arc capacities, source q and sink s is given.

	1	2	3
1	q+10	q+4	q+4
2	1+6	—	—
3	—	q+7	—
4	—	3+7	—
s	2+6	4+7	—
Δ	6	7	0
Σ	6	13	13



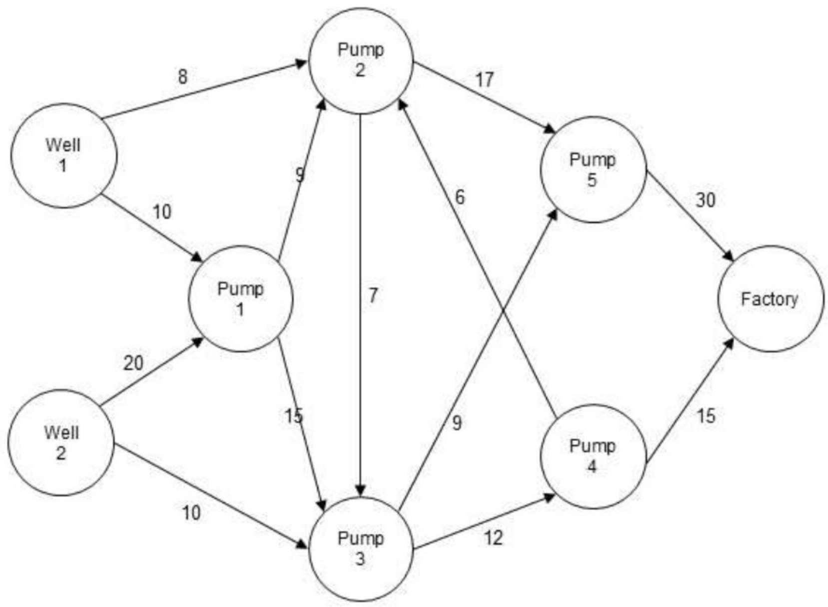
a) Modify the graph so that you are able to determine its maximum flow by using the Ford-Fulkerson algorithm.

b) Determine a feasible (not necessarily optimal) flow from the source to the sink. Start out from that flow and perform as many iterations of the Ford-Fulkerson algorithm as necessary to determine the maximum flow from the source to the sink.

c) Determine the cut with the minimum capacity. Mark the cut in the graph and determine its capacity.

Question 119

Munich's municipal services run two water wells at the maximum flow rates of 15 litres per second (Well 1) and 25 litres per second (Well 2). The water is pumped to a factory via a system of water pipes and pumps. The following graph depicts this system and the maximum capacities of the water pipes (in litres per second).



a) Modify the graph so that you are able to determine its maximum flow by using the Ford-Fulkerson algorithm.

b) Determine a feasible (not necessarily optimal) flow from both wells to the factory.

c) Start out from the feasible flow and perform as many iterations of the Ford-Fulkerson algorithm as necessary to determine the maximum flow to the factory.

Question 120

Consider the problem statement of question 119 again.

a) Explain the following terms:

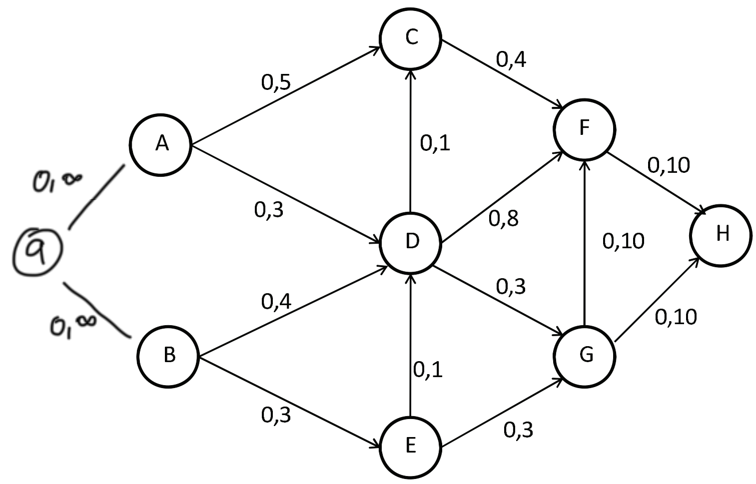
  - Cut
  - Capacity of a cut

b) Determine the cut with the minimum capacity.

c) Give an interpretation of the results.

Question 121

Consider the following network with arc capacities, two sources A and B, and sink H.



a) Modify the graph so that you are able to determine its maximum flow by using the Ford-Fulkerson algorithm.

b) Determine a feasible (not necessarily optimal) flow from the two sources to the sink. → Just set everything to 0

c) Start out from the feasible flow and perform as many iterations of the Ford-Fulkerson algorithm as necessary to determine the maximum flow to the the sink.

d) Which water pipes are necessary to retain their maximum capacities, so that the flow determined in b) is not changed?

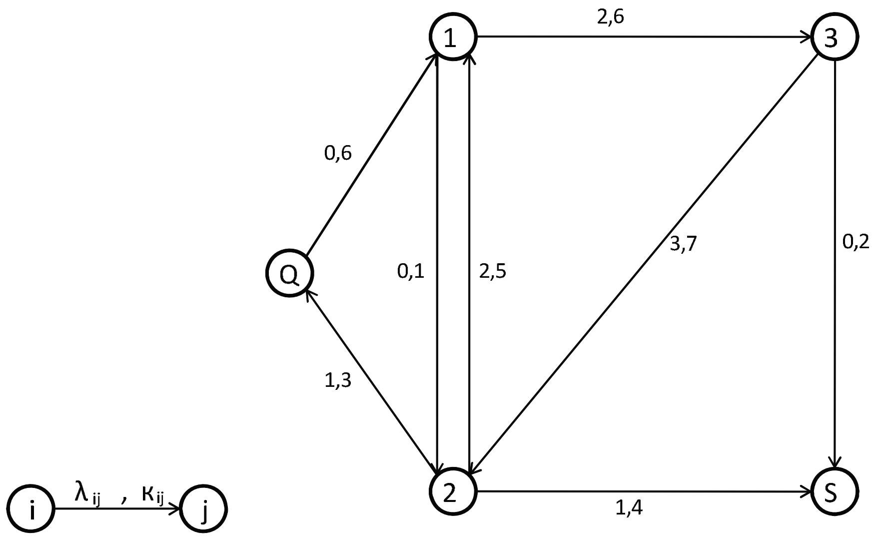
Question 122

Consider the following network with arc capacities, source Q and sink S.

a) Determine a feasible (not necessarily optimal) flow from the source to the sink.

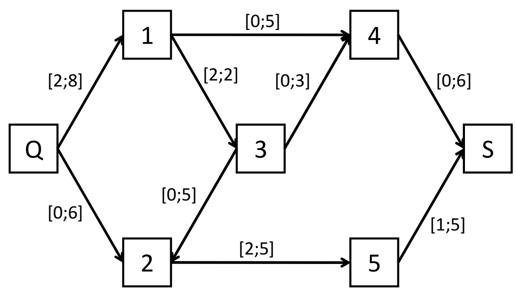
b) Start out from the feasible flow and perform as many iterations of the Ford-Fulkerson algorithm as necessary to determine the maximum flow to the sink.

c) The maximum capacity (minimum capacity) of which water pipe must not be reduced (increased) so that the determined flow of sub task b) is not changed.



Question 123

Consider the following network with arc capacities, source Q and sink S.



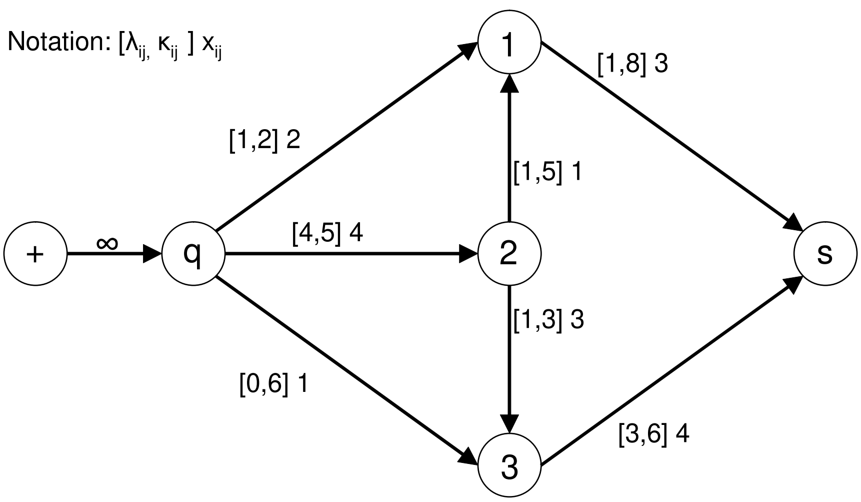
a) Modify the graph so that you are able to determine its maximum flow by using the Ford-Fulkerson algorithm.

b) Start out from the given feasible flow and perform as many iterations of the Ford-Fulkerson algorithm as necessary to determine the maximum flow to the sink.

c) Determine the cut with the minimum capacity. Mark the cut in the graph and determine its capacity.

Question 124

Consider the following directed graph with arc capacities λ_{ij} and a feasible flow K_{ij}.



a) Record the given flow in the following table.

b) Determine the maximum flow from source q to sink s by using the Ford-Fulkerson algorithm.

		Iteration
Node	Feasible Flow	
q		
1		
2		
3		
s		
Δ $\sum$		

c) Determine the set of nodes V_q and V_s of a cut with the minimum capacity. Mark the cut in the graph and determine its capacity.

5 Dynamic Programming

Question 125

You are the manager of a research department and have deployed 3 research teams to work on different technical solutions independently. You wish to hire just 2 new scientists and divide them to the three teams to minimize the probability that all teams fail. The following table presents the probability of failure for each team if k additional scientists are hired.

z_2	x_3	z_3	$f(z_3,x_3)$	$F^*(z_3)$	$F(z_2)$
0	0	0	0.8	1	0.8*
1	1	0	0.5	1	0.5*
2	2	0	0.3	1	0.3*

z_1	x_2	z_2	$f(z_2,x_2)$	$F^*(x_2)$	$F(z_1)$
2	0	2	0.6	0.3	0.18 -
2	1	1	0.4	0.5	0.2 -
1	0	1	0.6	0.5	0.3 *
2	2	0	0.2	0.8	0.16 *
1	1	0	0.4	0.8	0.32 -
0	0	0	0.6	0.8	0.48 *

k	Team 1	Team 2	Team 3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30

z_0	x_1	z_1	$f(z_1,x_1)$	$F^*(z_1)$	$F(z_0)$	
2	0	2	0.4	0.16	0.064	
2	1	1	0.2	0.3	0.06	*
2	2	0	0.15	0.48	0.072	

$x_1^* = 1$
$x_2^* = 0$
$x_3^* = 1$

a) Define each component and variable of a dynamic program to solve the problem.

b) Determine an optimal distribution of the two scientists using dynamic programming. Specify the number of scientists to be dedicated to each team and the probability that each team is successful.

Question 126

Hans wants to invest his savings of 10 in three investment alternatives, but he can choose each alternative only once. Forward recursion

z_1	x_1	z_0	$f(x_1)$	$F^*(z_0)$	$F(z_1,x_1)$	$F^*(z_1)$
0	0	0	0	0	0	*
3	1	0	19	0	19	*

z_2	x_2	z_1	$f(x_2)$	$F^*(z_1)$	$F(z_2,x_2)$	$F^*(z_2)$
0	0	0	0	0	0	*
3	0	3	0	19	19	*
4	1	0	20	0	20	*
7	1	3	20	19	39	*

Investment i	Capital Value c_i	Expenses a_i
1	19	3
2	20	4
3	25	5

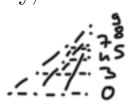
z_3	x_3	z_2	$f(x_3)$	$F^*(z_2)$	$F(z_3,x_3)$	$F^*(z_3)$
0	0	0	0	0	0	*
3	0	3	0	19	19	*
4	0	4	0	20	20	*
7	0	7	0	39	39	*
5	1	0	25	0	25	*
8	1	3	25	19	44	*
9	1	4	25	20	45 max	*
~~12~~	~~1~~	~~7~~	–	–		

$\rightarrow 1 x_3; 1 x_2; 0 x_1$

a) Determine the optimal investment policy. Specify which investments he should make with his achieved capital value and expenses.

b) Explain why you do not consider certain states in stage $i = 1$. Specify which states are not considered.

c) Draw a transition diagram. Mark the optimal policy, all subproblems, all subpolicies and all dominant subpolicies.



Question 127

Heinz Hectic has only 11 days left to prepare for 4 chapters to pass the "Management Science" exam. He wants to use this time as effectively as possible. While not to miss any chapter, he needs to spend at least ≥ two days and at most ≤ five days on each chapter and only one chapter per day. The exam result depends on the learning effort in the number of learning days. The following table presents the estimated points earned for each chapter and learning effort.

z_1	x_1	z_0	$f(x_1)$	$F^*(z_0)$	$F(z_1)$	opt?
2	2	0	5	0	5	*
3	3	0	7	0	7	*
4	4	0	7	0	7	*
5	5	0	9	0	9	*

z_2	x_2	z_1	$f(x_2)$	$F^*(z_1)$	$F(z_2)$	opt?
4	2	2	4	5	9	*
5	2	3	4	7	11	*
6	2	4	4	7	11	–
7	2	5	4	9	13	–
5	3	2	6	5	11	–
6	3	3	6	7	13	*
7	3	4	6	7	13	–
6	4	2	8	5	13	–
7	4	3	8	7	15	*
7	5	2	9	5	14	–

$z_2 = 4,5,6,7$

$z_3 \leq 9$

Number of days	Decision Theory z_1	Linear Programming z_2	Integer Programming z_3	Network Theory z_4
2	5	4	6	5
3	7	6	7	5
4	7	8	8	8
5	9	9	9	8

← 4+ days needed for z_3, z_4 → filtered for feasible states

z_3

z_3	x_3	z_2	$f(x_3)$	$F^*(z_2)$	$F(z_3)$	opt?
6	2	4	6	9	15	*
7	2	5	6	11	17	*
8	2	6	6	13	19	*
9	2	7	6	15	21	*
7	3	4	7	9	16	–
8	3	5	7	11	18	–
9	3	6	7	13	20	–
8	4	4	8	9	17	–
9	4	5	8	11	19	–
9	5	4	9	9	18	–

Z_4

z_4	x_4	z_3	$f(x_4)$	$F^*(z_3)$	$F(z_4)$	opt?
8	2	6	5	15	20	*
9	2	7	5	17	22	*
10	2	8	5	19	24	*
11	2	9	5	21	26	* ← optimum!
9	3	6	5	15	20	–
10	3	7	5	17	22	–
11	3	8	5	19	24	–
10	4	6	8	15	23	–
11	4	7	8	17	25	–
11	5	6	8	15	23	–

$x_4 = 2$
$x_3 = 2$
$x_2 = 4$
$x_1 = 3$

Heinz Hectic wants to achieve as many points as possible by allocating the 11 learning days to the 4 chapters effectively.

a) Define each component and variable of a dynamic program to solve the problem.

b) Determine an optimal allocation of the 11 learning days using dynamic programming. Specify the number of learning days allocated to each chapter and the maximum points he will earn in the exam.

Question 128

In the next three weeks a company needs the following amount of raw material:

z_2	x_3	z_3	$f(z_3,x_3)$	$F(z_3)^*$	$F(z_2)$	
0	40	0	150	0	150	*
40	0	0	200	0	200	*

z_1	x_2	z_2	$f(z_2,x_2)$	$F(z_2)^*$	$F(z_1)$	
20	0	0	100	150	250	*
0	20	0	150	150	300	*
60	0	40	300	200	500	*
0	60	40	150	200	350	-

Week	1	2	3
Demand	50	20	40

z_0	x_1	z_1	$f(z_1,x_1)$	$F^*(z_1)$	$F(z_0)$	
0	50	0	150	300	450	
0	70	20	150	250	400	*
0	110	60	150	500	650	

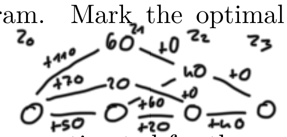
Optimal:
0→20→0→0
70,0,40
Total cost = 400

Raw material, that is not used in the current week, will cost 5 Euro of storage per week per unit. The fixed ordering cost is 150 Euro. Assume that at the beginning of the planning period the storage is empty and the ordered amount will be available immediately (No delivery lead time).

a) Define each component and variable of a dynamic program to solve the problem.

b) Determine an optimal order policy using dynamic programming. Specify the amount of the raw material to be ordered in each week and the total costs.

c) Draw a transition diagram. Mark the optimal policy, all subproblems, all subpolicies and all dominant subpolicies.



Question 129

The demands of a component are estimated for the next five months. Components, that are not used in the current month, will cost 6 Euro of storage per month and 1000 units. The fixed ordering cost is 200 Euro.

Month	1	2	3	4	5
Demand	6,000	4,000	22,000	27,000	8,000

Assume that at the beginning of the planning period the storage is empty and the ordered amount will be available immediately (No delivery lead time).

a) Define each component and variable of a dynamic program to solve the problem.

b) Determine an optimal order policy using dynamic programming. Specify the number of the components to be ordered in each month and the total costs.

Question 130

A financial service company invested 10,000 Euro of his clients in the fixed-income securities with an interest coupon of 10% per period. The interest is paid without compounding at the end of the sixth period. Through the sale of the securities, the company has to ensure the paybacks to their clients due at the beginning of each period.

Period	1	2	3	4	5	6
Payback	1,500	2,500	500	1,000	3,000	1,500

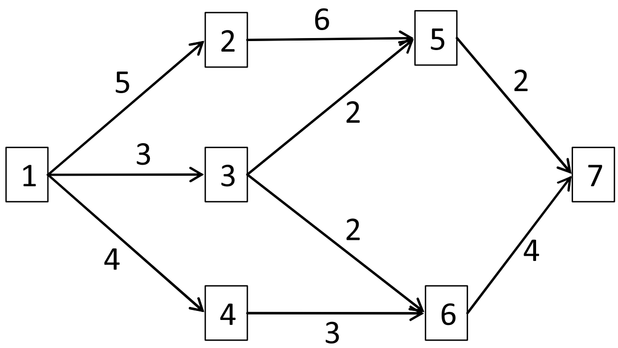
A sale order of securities costs 250 Euro independent of the financial value. The fixed order cost has to be paid at the end of the sixth period. The company wants to determine an optimal divestment policy to minimize the total costs (= opportunity costs for interest revenues plus fixed costs).

a) Define each component and variable of a dynamic program to solve the problem.

b) Determine an optimal divestment policy using dynamic programming. Specify when and in which amount the securities will be sold and the minimal total costs.

Question 131

In this question the shortest-path-problem from node 1 to node 7 should be solved by using dynamic programming. In this regard consider the following graph

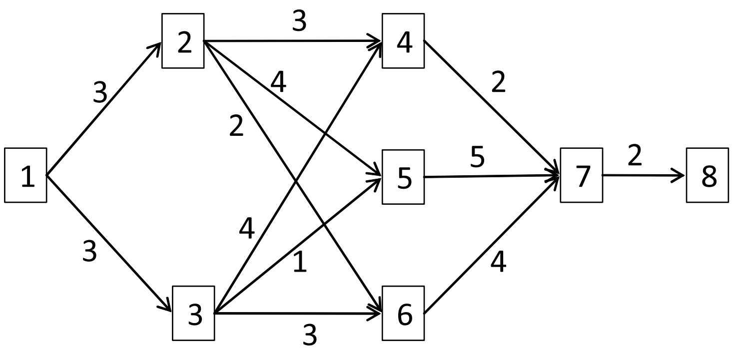


a) Define each component and variable of a dynamic program to solve the problem. State the component for each node.

b) Determine the shortest path from node 1 to node 7 by using dynamic programming. State the shortest path and its length.

Question 132

In this question the shortest-path-problem from node 1 to node 8 should be solved by using dynamic programming. In this regard consider the following graph



a) Define each component and variable of a dynamic program to solve the problem. State the component for each node.

b) Determine the shortest path from node 1 to node 8 by using dynamic programming. State the shortest path and its length