

Solution 3: Production and Supply

Problem 1 (Production Function)

(a) Positive and decreasing marginal products of both inputs

(i) Marginal Product of Labor:

$$\frac{\partial F(L, K)}{\partial L} = \alpha L^{\alpha-1} K^\beta > 0, \quad \frac{\partial^2 F(L, K)}{\partial L^2} = (\alpha - 1) \alpha L^{\alpha-2} K^\beta < 0$$

(ii) Marginal Product of Capital:

$$\frac{\partial F(L, K)}{\partial K} = \beta L^\alpha K^{\beta-1} > 0, \quad \frac{\partial^2 F(L, K)}{\partial K^2} = (\beta - 1) \beta L^\alpha K^{\beta-2} < 0$$

(b) Returns to Scale: Provided that $L > 0$ and $K > 0$, a multiplication of both inputs by a constant λ implies a multiplication of output by $\lambda^{\alpha+\beta}$.

$$F(\lambda L, \lambda K) = (\lambda L)^\alpha \cdot (\lambda K)^\beta = \lambda^{\alpha+\beta} L^\alpha K^\beta = \lambda^{\alpha+\beta} F(L, K)$$

Thus, the production function exhibits

- (i) constant returns to scale if $\alpha + \beta = 1$,
- (ii) increasing returns to scale if $\alpha + \beta > 1$,
- (iii) decreasing returns to scale if $\alpha + \beta < 1$.

(c) Strictly Convex Isoquants: Solving the production function for K yields the isoquant for a given output level q .

$$K = q^{\frac{1}{\beta}} \cdot L^{-\frac{\alpha}{\beta}}$$

The first partial derivative of the isoquant with respect to L is

$$\frac{\partial K}{\partial L} = -\frac{\alpha}{\beta} \cdot q^{\frac{1}{\beta}} \cdot L^{-\frac{\alpha+\beta}{\beta}} < 0,$$

while the second partial derivative of the isoquant with respect to L is

$$\frac{\partial^2 K}{\partial L^2} = \frac{\alpha^2 + \alpha\beta}{\beta^2} \cdot q^{\frac{1}{\beta}} \cdot L^{-\frac{\alpha+2\beta}{\beta}} > 0.$$

Hence, the isoquant has a negative slope that gets less negative, i.e. flatter, as L increases. Consequently, the isoquants are strictly convex.

Problem 2 (*Cost Minimization*)

(a) Optimal input bundle

- (i) The optimal input bundle must be located on the isoquant representing the given output level.

$$q = F(L, K) = (L \cdot K)^{\frac{1}{2}}$$

- (ii) In any interior solution, the optimal input bundle must be located where the isoquant is tangent to the lowest attainable isocost line, i.e. the MRTS must equal the input price ratio.

$$\underbrace{\frac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}}}_{\text{MRTS}_{L,K}} = \frac{w}{r} \Leftrightarrow \frac{K}{L} = \frac{w}{r} \Leftrightarrow K = \frac{w}{r}L \Leftrightarrow L = \frac{r}{w}K$$

Substituting (ii) into (i) yields

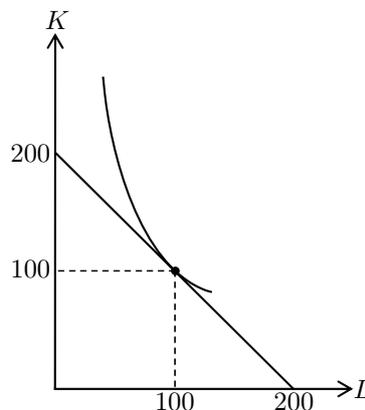
$$L = q \left(\frac{r}{w}\right)^{\frac{1}{2}}, \quad K = q \left(\frac{w}{r}\right)^{\frac{1}{2}}.$$

Given the output $q = 100$ and input prices $w = 2.5$ and $r = 2.5$, the optimal input bundle is

$$L = 100, \quad K = 100,$$

and the corresponding input costs are

$$c = wL + rK = 2.5 \cdot 100 + 2.5 \cdot 100 = 500.$$



Optimal Input Employment

(b) Now, the same output $q = 100$ shall be produced given a higher wage rate for labor $w' = 10$ and an unchanged rental rate for capital $r = 2.5$.

(i) At the new input price ratio, the input bundle $L = 100$ and $K = 100$ no longer minimizes costs. It causes input costs of

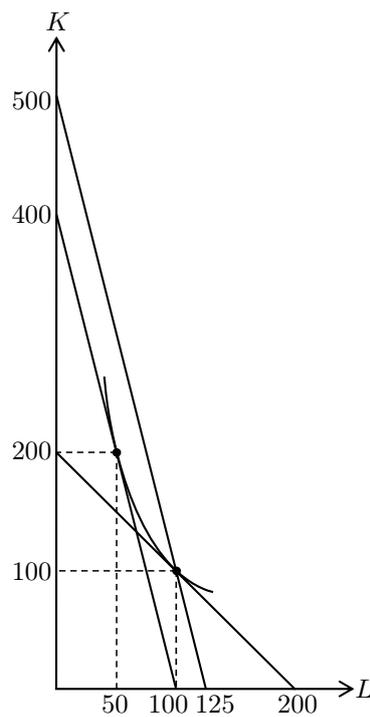
$$c = w'L + rK = 10 \cdot 100 + 2.5 \cdot 100 = 1,250.$$

(ii) The new cost minimizing input bundle is

$$L' = 50, \quad K' = 200,$$

and the corresponding input costs are

$$c' = w'L' + rK' = 10 \cdot 50 + 2.5 \cdot 200 = 1,000.$$



(Optimal) Input Employment

Problems 3-5 (Cost Minimization)

Problem 3

Provided that $L > 0$ and $K > 0$, a multiplication of both inputs by 4 implies a multiplication of output by 2.

$$F(4L, 4K) = (4L \cdot 4K)^{\frac{1}{4}} = 2(L \cdot K)^{\frac{1}{4}} = 2F(L, K)$$

\Rightarrow (B) is correct.

Optimal input bundle

- (i) The optimal input bundle must be located on the isoquant representing the given output level.

$$q = F(L, K) = (L \cdot K)^{\frac{1}{4}}$$

- (ii) In any interior solution, the optimal input bundle must be located where the isoquant is tangent to the lowest attainable isocost line, i.e. the MRTS must equal the input price ratio.

$$\underbrace{\frac{\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{4}}}{\frac{1}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}}}}_{\text{MRTS}_{L,K}} = \frac{w}{r} \Leftrightarrow \frac{K}{L} = \frac{w}{r} \Leftrightarrow K = \frac{w}{r}L \Leftrightarrow L = \frac{r}{w}K$$

Substituting (ii) into (i) yields

$$L = q^2 \left(\frac{r}{w}\right)^{\frac{1}{2}}, \quad K = q^2 \left(\frac{w}{r}\right)^{\frac{1}{2}}.$$

Thus, variable costs (minimum input costs as a function of output) are

$$c(q) = w \cdot q^2 \left(\frac{r}{w}\right)^{\frac{1}{2}} + r \cdot q^2 \left(\frac{w}{r}\right)^{\frac{1}{2}} = 2(wr)^{\frac{1}{2}}q^2.$$

Problem 4

If the wage rate for labor is given by $w = 16$, and the rental rate for capital is given by $r = 4$, variable costs are $c(q) = 16q^2$.

\Rightarrow (D) is correct.

Problem 5

If the wage rate for labor is given by $w = 18$, and the rental rate for capital is given by $r = \frac{1}{2}$, variable costs are $c(q) = 6q^2$ and marginal costs are $MC(q) = 12q$.

\Rightarrow (D) is correct.

Problems 6-8 (Profit Maximization)

Marginal Costs: $MC(q) = 4q$

Average Total Costs: $AC(q) = \frac{200}{q} + 2q, \quad \forall q > 0$

Average Variable Costs: $ac(q) = 2q$

Problem 6

For $q = 20$, marginal costs are higher than average total costs.

$$MC(20) = 80 > 50 = AC(20)$$

\Rightarrow (B) is correct.

Problem 7

If the firm produces $q > 0$, profit maximization requires $p = MC(q)$. For $p = 20$, this yields

$$20 = 4q \quad \Leftrightarrow \quad q = 5.$$

In the short run, the firm produces $q = 5$, because the price is higher than average variable costs at $q = 5$.

$$p = 20 > 10 = ac(5)$$

In the long run, the firm produces $q = 0$, because the price is lower than average total costs at $q = 5$.

$$p = 20 < 50 = AC(5)$$

Thus, if $p = 20$, the firm's supply is 5 in the short run and 0 in the long run.

\Rightarrow (B) is correct.

Problem 8

The threshold price, above which the firm's supply is $q > 0$ in the long run, corresponds to the quantity for which marginal costs equal average total costs, i.e. where average total costs reach their minimum.

$$MC(q) = 4q = \frac{200}{q} + 2q = AC(q) \quad \Rightarrow \quad q = 10$$

Thus, the threshold price is

$$p = MC(10) = 40.$$

\Rightarrow (D) is correct.